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## **CONSTITUTIVE EQUATIONS ON MOVEMENT OF WATER AND AIR IN SOILS ON BASIS OF THEORY OF POROUS MEDIA**

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### **Abstract**

On basis of theory of porous media new constitutive equations on movement of two fluids in a porous media are developed. These constitutive equations and so the system of governing differential equations are useful to describe the classical 'unsaturated flow'-problems as well as typical two-phase flow problems. They can be generally extended to flow in porous media with changing porosity or three-phase flow.

Classical models to describe two-phase flow, basing on Darcy's, Richards' and van Genuchten's ideas are compared with the new developed equations. These classical concepts are recognized to be limited to special cases and to be not helpful to understand mechanical behaviour of fluid flow in general.

### **Introduction**

In geotechnical, environmental and tunnelling engineering a lot of assignments need the quantitative description of ground water flow in soils. Flow of air in soils, appearing in air suction method for purification of highly polluted soils or tunnelling under compressed air, grows on interest more and more. Hereby in- and outflowing rates of the fluids are most of interest.

By describing water and air flow in soils we have to distinguish between one-phase and two-phase flow in general. If there is only one-phase flow, calculations of flow are relatively simple and also described by a lot of authors [4] already.

Though, if there is flow of water *and* air in soil, several interactions occur which have not been described and calculated in a satisfactory way, yet. A typical field of application of two-phase flow in soils is tunnelling under compressed air.

A research program on movement of water and air in soils is supported by the German Research Council (DFG). Equations are formulated here with rational mechanics [15]. On basis of the theory of porous media and the theory of mixtures [3] new constitutive equations have to be formulated for pressures of fluids including capillary effects and the interaction terms like the so-called 'seepage forces'.

### **Simplification**

Some simplifications have to be made in this paper to concentrate on fundamental effects. First of all we will restrict the following discussion on fluid movement, thus the soil skeleton may be modelled as isotropic and rigid continuum. This paper will not deal with constitutive equations for the solid; they are not relevant for movement of fluids under this conditions.

The pores of the continuum are completely filled with the Newtonian fluids water and air. Further, it is helpful to consider the temperature as a constant. This assumption prevents heat conduction and heat flux between the solid and fluid phases; water should appear only in liquid phase. Phase transitions between the fluids and the fluids and the solid phase will be neglected.

### **Volume Fraction**

We consider a macroscopic model in which the constituents are statistically distributed over the control space formed by the porous media [2]. This means that each constituent  $i$  takes up a certain fraction of the volume of the control space - together with the other constituents. Here the constituents are water (index F) as wetting phase and gas (index G) as nonwetting phase. The solid is indicated with S.

Each point of the continuum gets the fractions  $n_i$  from the partial volumes  $dv_i$  and the total volume  $dv$  of a representative element.

$$dv_i = n_i \cdot dv \quad (1)$$

Because the control space is completely filled with the constituents  $i$  the condition of saturation (2) is valid.

$$n_S + n_F + n_G = 1 \quad (2)$$

Within the theory of mixtures the partial or bulk density  $\rho_i$  can be defined as the mass of a constituent  $i$  per unit of the representative volume [2]. This relation can be written with the volume fraction  $n_i$  and the real density  $\rho_{iR}$  as:

$$\rho_i = n_i \cdot \rho_{iR} \quad (3)$$

The partial pressure for a constituent  $i$  can be determined in the same way.

### Balance Equations

The balance equations have to be fulfilled for the whole mixture as well as for every constituent. Therefore it is possible to formulate physical principles for the whole mixture in adding the single constituents [3].

By assuming that phase transitions are negligible the balance of mass has the following local form, by  $i \in \{F,G\}$ , (see [3]):

$$\frac{\partial \rho_i}{\partial t} + \text{grad} \rho_i \cdot \dot{\mathbf{x}}_i + \rho_i \cdot \text{div} \dot{\mathbf{x}}_i = 0 \quad (4)$$

The balance equations of momentum of the fluids, see [3], describe relations between partial stresses, accelerations  $\ddot{\mathbf{x}}_i$  and the partial density  $\rho_i$  of each constituent. Here, we get:

$$\text{grad}(-p_i) + \rho_i \cdot (\mathbf{b} - \ddot{\mathbf{x}}_i) + \hat{\mathbf{s}}_i = \mathbf{0} \quad (5)$$

The external body force density  $\mathbf{b}$  appears in general because it has for all constituents  $i$  the same value. On surface of earth the parameter  $\mathbf{b}$  is equal to the earth acceleration  $\mathbf{g}$ .

For simplicity, in (5) the divergence of the stress tensor ( $\text{div} \underline{T}_i$ ) has already been replaced by  $\text{grad}(-p_i)$  [6]. The partial pressure  $p_i$  is defined as positive, here. Within balance equation of momentum, even for Newtonian fluids it is allowed to consider the stress tensor as hydrostatic [1]. That means in divergence term the deviatoric components are negligible for fluid movement.

Because the balance equation of momentum for the whole mixture has to be fulfilled, the terms of interaction between the constituents  $\hat{\mathbf{s}}_i$  from (5) are coupled:

$$\hat{\mathbf{s}}_F + \hat{\mathbf{s}}_G + \hat{\mathbf{s}}_S = \mathbf{0} \quad (6)$$

From the balance of moment of momentum [2], [6] follows the symmetry of the partial stress tensor. This information is already included in (5).

As a consequence of the isothermal model the conservation law of the balance of energy also cannot give new information.

### Pressure Relations

The positive partial pressure for incompressible fluids is determined after (3) by a product of the volume fraction and the real density  $\rho_{iR}$  as a constant. Therefore the constitutive equations, however, cannot deliver the pressure relations and (4) reduces itself to a equation of volume balance.

For compressible fluids the partial pressure can be determined by Boyle-Mariotte's law. The constitutive relations have the form [8], [10]:

$$-p_G = -\rho_G \cdot R \cdot T \quad (7)$$

with  $T$  as absolute temperature,  $\rho_G$  as partial density and  $R$  as gas constant.

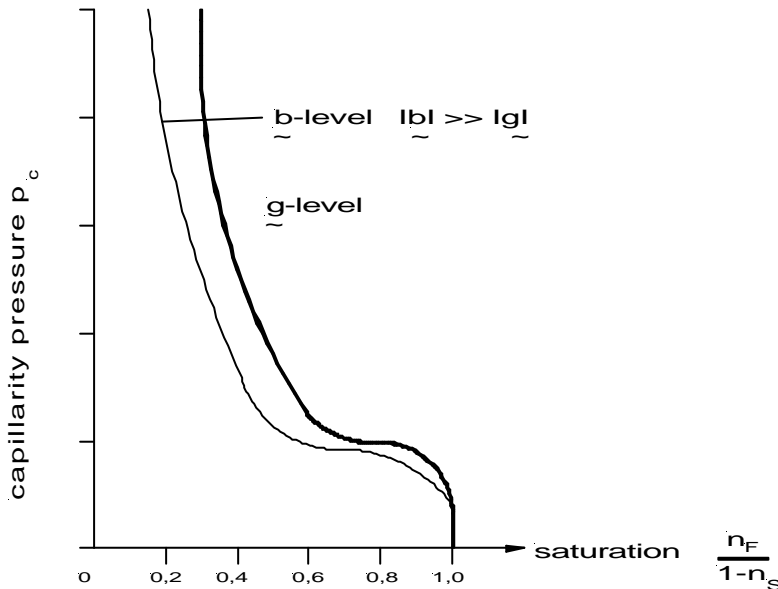
The fluid-pressure relation in the static state, however, is given by the capillarity function, using the partial pressures  $p_G$  and  $p_F$ , introduced above, as well as the water saturation.

So the capillarity function can be formulated in general in the following form:

$$F\left(p_F, p_G, \frac{n_F}{1-n_S}\right) = 0 \quad (8)$$

From basic mechanics follows that this constitutive function (8) *must not* depend on the acceleration field  $\mathbf{b}$ , because this is an external influence. In our concept, basing on the theory of porous media, the acceleration  $\mathbf{b}$  will be placed exclusively within the balance equation of momentum (5).

For a general approach it is necessary to avoid a capillarity function which depends on  $\mathbf{b}$ . The classical capillarity function *does* depend on the earth acceleration  $\mathbf{g}$  - that is observed in experiments [14]. Fig. 1 shows two *different* capillarity curves for the *same* soil. One of them bases on dewatering experiments under  $\mathbf{g}$ -level and is fitted with van Genuchten's [9] equation. The values from the other curve are gained from a centrifuge dewatering experiment with a different  $\mathbf{b}$ -level and deviates specially at low saturation.



**Fig. 1** Effect of the acceleration field on the capillarity curves [14]

Consequently van Genuchten's [9] equation and so the other classical capillarity curves cannot be used to formulate constitutive equations.

### Interaction Forces

The interaction forces  $\hat{s}_i$  in (5) and (6) are decisive for the description of the fluid flow through porous media. The forces have to be determined by constitutive equations. Both equations (9), (10) have three summed terms. The first represents the effect of spatial variation of the porosity. The second and the third term show difference velocities of the constituents multiplied by resistivities  $R_{ij}$ .

$$\hat{s}_F = \frac{p_F}{n_F} \cdot \text{grad } n_F - R_{FS} \cdot (\dot{\mathbf{x}}_F - \dot{\mathbf{x}}_S) - R_{FG} \cdot (\dot{\mathbf{x}}_F - \dot{\mathbf{x}}_G) \quad (9)$$

$$\hat{s}_G = \frac{p_G}{n_G} \cdot \text{grad } n_G - R_{GF} \cdot (\dot{\mathbf{x}}_G - \dot{\mathbf{x}}_F) - R_{GS} \cdot (\dot{\mathbf{x}}_G - \dot{\mathbf{x}}_S) \quad (10)$$

The resistivities  $R_{ij}$  depend on the fluid saturation. A special case of these interaction forces appears as seepage forces in classical soil mechanics. In combination with (6) these equations lead to the respective expression  $\hat{s}_s$  for the solid.

Expressions like the one presented above had already been introduced by [7], [13], e.g. These formulations are similar to those of Chen [5]. The approach from (9) and (10), however, seems to be more meaningful from physical point of view.

From evaluation of (6) we obtain a relation for the two water-gas-interaction resistivities:

$$R_{FG} = R_{GF} \quad (11)$$

## Classical Formulas

In soil mechanics Richard's generalisation of Darcy's law  $\mathbf{v}_i = K_i \cdot \mathbf{i}_i$  is a widely used description for two-phase flow in porous media, where  $K_i$  is a function of the respective fluid saturation.  $K_i$  contains a constant (full saturation for i) term and a multiplied reduction factor  $k_{ri}$  which depends on saturation of the phase i [1], [8], [10].

With some specifications the balance equation of momentum (5) can be transformed to Richard's generalisation of Darcy's law. First of all we assume the acceleration  $\mathbf{b}$  to be equal to  $\mathbf{g}$  and the acceleration of the fluid to be negligible. Furthermore, we postulate  $R_{GF} \equiv R_{FG} \equiv 0$  - in accordance with some literature, where it is argued that the terms containing  $R_{FG}$  and  $R_{GF}$  were relatively small and could be neglected [1]. So we get Richard's generalisation of Darcy's law:

$$\underbrace{n_i \cdot (\dot{\mathbf{x}}_i - \dot{\mathbf{x}}_S)}_{\mathbf{v}_i} = \underbrace{k_{ri} \cdot \frac{k \cdot \rho_{iR} \cdot |\mathbf{g}|}{\mu_i}}_{K_i} \cdot \underbrace{\left( -\frac{1}{\rho_{iR} \cdot |\mathbf{g}|} \cdot \text{grad } p_{iR} + \frac{1}{|\mathbf{g}|} \cdot \mathbf{g} \right)}_{\mathbf{i}_i} \quad (12)$$

As a result the resistivity terms in (9) and (10) can be expressed as follows:

$$R_{FS} = \frac{\mu_F \cdot n_F^2}{k \cdot k_{rF}} \quad R_{GS} = \frac{\mu_G \cdot n_G^2}{k \cdot k_{rG}} \quad R_{GF} = 0 \quad R_{FG} = 0 \quad (13 \text{ a-d})$$

with  $k$  as intrinsic permeability of the soil and  $\mu_i$  as viscosity of the fluid phase i. In the special case of one-phase flow the relative permeability  $k_{ri}$  becomes 1 for the existing fluid [10].

In Richard's concept each of the two fluids air and water occupies a part of the pore volume; this is a second relation between air and water besides the capillarity function. There are no interactions of forces between the two fluids as can be seen from (9) and (10) with (13 c) and (13 d).

A more sophisticated generalisation of Darcy's law is introduced by [11], [16] and others. They use an approach which couples the velocity of each fluid with 'hydraulic gradients' of both fluids. It can be formulated as follows:

$$\begin{bmatrix} \mathbf{v}_F \\ \mathbf{v}_G \end{bmatrix} = \begin{bmatrix} n_F \cdot (\dot{\mathbf{x}}_F - \dot{\mathbf{x}}_S) \\ n_G \cdot (\dot{\mathbf{x}}_G - \dot{\mathbf{x}}_S) \end{bmatrix} = \begin{bmatrix} -k_{FF} & -k_{FG} \\ -k_{GF} & -k_{GG} \end{bmatrix} \cdot \begin{bmatrix} \text{grad } p_{FR} - \rho_{FR} \cdot \mathbf{b} \\ \text{grad } p_{GR} - \rho_{GR} \cdot \mathbf{b} \end{bmatrix} \quad (14)$$

The  $k$ -matrix is called 'mobility matrix' and contains  $k_{ij}$  coefficients, which are defined after (15).

$$k_{ij} = \frac{k \cdot k_{rij}}{\mu_j} \quad (15)$$

Again,  $k$  is the intrinsic permeability of the soil and  $\mu_j$  the viscosity of fluid  $j$ .  $k_{ij}$  are coefficients of the relative permeability matrix and depend on fluid saturation. From Onsager's reciprocal relation follows:  $k_{FG}=k_{GF}$  [11].

This generalisation of Darcy's takes the effect of co- and counter-current flow of two fluids into account. Co-current means that both fluids move in the same direction. This will increase the velocity of the slower fluid. The counter-current flow describes a two-phase flow with opposite flow directions. The result of this type of flow is an additional drag of both fluid flows.

Similar to Richard's generalisation of Darcy's law, resistivity terms according to (9) and (10) can be developed for this approach. After elementary mathematical transformations the terms are [12]:

$$R_{FS} = \frac{k_{GG} \cdot n_F^2 - k_{FG} \cdot n_F \cdot n_G}{k_{FF} \cdot k_{GG} - k_{FG} \cdot k_{GF}} \quad (16)$$

$$R_{GS} = \frac{k_{FF} \cdot n_G^2 - k_{GF} \cdot n_G \cdot n_F}{k_{FF} \cdot k_{GG} - k_{FG} \cdot k_{GF}} \quad (17)$$

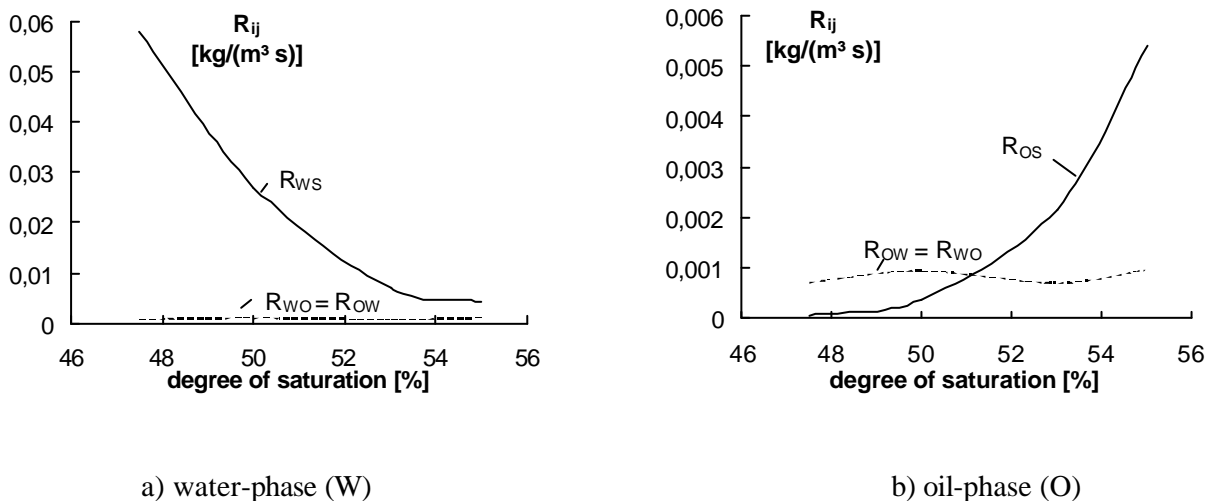
$$R_{GF} = R_{FG} = \frac{n_F \cdot n_G \cdot k_{FG}}{k_{FF} \cdot k_{GG} - k_{FG} \cdot k_{GF}} \quad (18)$$

It can be seen easily that these equations contain Richard's generalisation of Darcy's law (12) as a special case. Setting the nondiagonal terms  $k_{FG}$  and  $k_{GF}$  to 0, we get (13a-d).

From (18), and so from the balance equation of momentum of the mixture (6), follows that  $k_{FG}=k_{GF}$  and so the Onsager relation is valid here - but, however, not necessary. It can be deduced only with basic principles of rational mechanics. The here presented equations base on rational mechanics and contain the most important ideas from classical and contemporary laws for two-phase flow.

The resistivities  $R_{ij}$  depend on the degree of saturation and have to be investigated from experiments. Test results will be given in a separate paper.

For first estimations experimental results from literature are available. Kalaydjian [11] e.g., used oil as nonwetting and water as wetting phase. Evaluation of (16), (17) and (18) give a first impression about the magnitude of the resistivities measured in the tests of Kalaydjian.



**Fig. 2** Resistivity functions based on experimental results by Kalaydjian [11], see [12]

Fig. 2 shows that the nondiagonal terms  $R_{ij}$  may be of different meaning for the balance equations of the two fluids. On the one hand neglecting the nondiagonal term  $R_{WO}$  can be reasonable for flow of fluid W, since  $R_{WO}$  is small compared to  $R_{WS}$  for all degrees of saturation. But, on the other hand the disregard of the (identical) nondiagonal term  $R_{OW}$  would lead to a considerable fault in case of the oil phase. So, the above mentioned arguments

about insignificance of the nondiagonal terms cannot hold up in general. These terms may be important for a lot of assignments, e.g. air suction methods or tunnelling under compressed air.

## Conclusions

An evaluation of Richard's generalisation of Darcy's law and van Genuchten's capillarity function with rational mechanics shows that these theories need some simplifications which cause deficiencies under certain conditions. Richard's and van Genuchten's concepts, however, give reasonable results in some special cases. But the limits of these concepts for practical use are not clear and it would be very difficult to extend these models. Consequently, they are not appropriate to understand mechanical behaviour of fluid flow in general.

In opposition to these conventional and intuitive established models, the presented approach on basis of theory of porous media allows to describe the two-phase flow rationally. The model can be extended in various ways when necessary.

For practical purposes there is still some work to be done. So, the capillarity function (8) and the resistivity terms in (9) and (10) have to be determined in detail.

## Notations

$b$	external body force density [m/s <sup>2</sup> ]	$\hat{s}_i$	interaction term of the balance equation of momentum (phase i) [N/m <sup>3</sup> ]
$k$	intrinsic permeability of the medium [m <sup>2</sup> ]	$\dot{x}_i$	velocity of phase i [m/s]
$k_{ij}$	coefficients of mobility matrix [m <sup>3</sup> · s/kg]	$K$	Darcy conductivity [m/s]
$k_{rij}$	coefficients of relative permeability matrix [-]	$R_{ij}$	coefficients of resistivity [kg/(m <sup>3</sup> · s)]
$k_{ri}$	relative permeability of phase i [-]	$\rho_i$	partial density of phase i [kg/m <sup>3</sup> ]
$n_i$	volume fraction of phase i [-]	$\rho_{iR}$	real density of phase i [kg/m <sup>3</sup> ]
$p_i$	partial pressure of phase i [N/m <sup>2</sup> ]	$\mu_i$	viscosity of the fluid phase i [N· s/m <sup>2</sup> ]
$p_{Ri}$	real pressure of phase i [N/m <sup>2</sup> ]		

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