A direct solver for the least-squares spectral collocation system on rectangular elements for the Stokes equations

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Abstract

A least-squares spectral collocation scheme for the steady and unsteady Stokes equations is proposed. The original domain is decomposed into quadrilateral subelements and on the element interfaces continuity of the functions is enforced in the least-squares sense. The collocation conditions and the interface conditions lead to overdetermined systems. These systems are directly solved by QR decomposition of the underlying matrices. By numerical simulations it is shown that the direct method leads to better results than the approach with normal equations. Furthermore it is shown that the condition numbers can be reduced by introducing the Clenshaw-Curtis quadrature rule for imposing the average pressure to be zero.

Keywords: Stokes equations, least-squares, spectral collocation, direct solvers, condition numbers, improved stability

1 Introduction

Spectral methods (see, e.g., Canuto et al. [2], Gottlieb and Orszag [5], [15] or Deville et.al. [3]) employ global polynomials for the numerical solution of differential equations. Hence they give very accurate approximations for smooth solutions with relatively few degrees of freedom. For analytical data exponential convergence can be achieved. If one deals with problems with non-smooth solutions (e.g., discontinuities or layers) the usual (global) continuous spectral approach yields very poor approximation results. To avoid these difficulties the original domain has to be decomposed into several subdomains where jumps at the interfaces are allowed. Gerritsma and Proot showed in [6] the good performance of discontinuous least-squares spectral element methods. In [10] we extended the above approach to one-dimensional singular perturbation problems where the least-squares spectral collocation schemes lead to a stabilization. Heinrichs extended the in [11] proposed least-squares spectral collocation method to a triangular decomposition [13] of the original domain and achieved good numerical

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results. Here we extend the method to a decomposition in quadrilaterals of the original domain and apply these scheme to the two-dimensional Stokes equations. The collocation conditions together with the interface conditions lead to an overdetermined system that can be approximately solved by least-squares. The essential enhancements of the here introduced scheme is the increased accuracy because of the use of a direct solver. For the overdetermined system we compute the QR decomposition of the associated matrix and solve the system. Because of avoiding the normal equations we obtain linear systems of equations with dramatically reduced condition numbers and so round-off errors do not have such a big influence to the approximation results.
For the Stokes problem the velocity and the pressure cannot be approximated independently due to the well known Babuška - Brezzi condition. If the velocity and the pressure are approximated by polynomials of the same degree eight spurious modes are introduced which lead to an unstable system (see Bernardi, Canuto and Maday [1]). A well-known compatible approximating velocity-pressure pair is the so-called $\mathbb{P}_N \times \mathbb{P}_{N-2}$ formulation, see, e.g., Rønquist [21]. Heinrichs [7], [9] employed this technique for the splitting of the Stokes equations. There the velocity components are approximated by polynomials in $\mathbb{P}_N$ and the pressure by two degrees lower order polynomials in $\mathbb{P}_{N-2}$. The resulting discrete system constitutes a saddle point problem which is diffcult to solve numerically.
Least-squares techniques offer theoretical and numerical advantages over the classical methods. Spectral least-squares methods were first introduced by Gerritsma and Proot in [17], [18], [19]. Heinrichs investigated least-squares spectral collocation schemes in [11], [12], [13] that lead to symmetric and positive definite algebraic systems which circumvent the LBB stability condition. In summary, our approach has the following advantages:

- equal order interpolation polynomials can be employed
- it is possible to vary the polynomial order from element to element
- improved stability properties for singular perturbation problems [4], [10] and Stokes or the Navier-Stokes equations [11], [12], [13], [17], [18], [19]
- good performance in combination with a decomposition of the original domain
- direct and efficient iterative solvers for positive definite systems can be used
- implementation is straightforward.

The paper is organized as follows. In Section 2, the first-order formulation of the Stokes equations is introduced. In Section 3 we describe the least-squares spectral collocation scheme, specify the domain decomposition and demonstrate two strategies of avoiding the pressure constant. Section 4 shows the numerical simulations with the results for the steady Stokes equations in subsection 4.1 and for the unsteady one in subsection 4.2. Finally, a conclusion is presented.

2 The Stokes equations

In order to apply least-squares the Stokes problem is transformed into an equivalent first-order system of partial differential equations. This is accomplished by introducing the vorticity $\omega = \nabla \times \mathbf{u}$ as an auxiliary variable. By using the identity

$$\nabla \times \nabla \times \mathbf{u} = -\Delta \mathbf{u} + \nabla(\nabla \cdot \mathbf{u})$$

and the incompressibility constraint $\nabla \cdot \mathbf{u} = 0$ we obtain

$$\frac{\partial u}{\partial t} + \nu \nabla \times \omega + \nabla p = \mathbf{f} \quad \text{in } \Omega \ , \ t \in [0, t_{end}] \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \ , \ t \in [0, t_{end}] \tag{2}$$

$$\omega - \nabla \times \mathbf{u} = 0 \quad \text{in } \Omega \ , \ t \in [0, t_{end}] \tag{3}$$

where $\mathbf{u}^T = [u_1, u_2]$ denotes the velocity vector, p the pressure, $\mathbf{f}^T = [f_1, f_2]$ the forcing term and ν the kinematic viscosity. Here it is assumed that the density equals unity. We impose the average pressure to be zero; i.e.,

$$\int_\Omega p \, d\mathbf{x} = 0, \tag{4}$$

since the pressure is only determined up to a constant.
For the time integration we use a second-order BDF scheme (see, e.g., [7]): If Δt denotes the step size in t and the index $n+1$ indicates that the functions are evaluated at the time step $t_{n+1} = (n+1)\Delta t$, the approximation of $(\frac{\partial \mathbf{u}}{\partial t})^{n+1}$ can be written as

$$\frac{\frac{3}{2}\mathbf{u}^{n+1} - 2\mathbf{u}^n + \frac{1}{2}\mathbf{u}^{n-1}}{\Delta t}. \tag{5}$$

Spectral least-squares methods for the Stokes and Navier-Stokes equations were first introduced in [17], [18] and [19]. Heinrichs [11] first investigated least-squares spectral collocation schemes for the Navier-Stokes equations and extended these methods to a triangular decomposition of the domain combined with adaptive mesh refinement in reference [13]. Here we extend the least-squares spectral collocation scheme for the Stokes equations to a quadratic decomposition of the domain and investigate a direct solver (QR decomposition) for the discrete algebraic systems. Furthermore we study the condition numbers of the algebraic systems for different discretizations of the Stokes equations.

3 The least-squares spectral collocation scheme

For the spectral approximation we introduce the polynomial subspace

$$\mathbb{P}_N = \{\text{Polynomials of degree} \leq N \text{ in both variables } x_1, x_2\}.$$

Now all unknwon functions are approximated by polynomials of the same degree N, i.e., u_1, u_2, ω, p are approximated by interpolating polynomials u_1^N, u_2^N, ω^N, $p^N \in \mathbb{P}_N$. Furthermore we have to introduce the standard Chebyshev Gauss-Lobatto collocation nodes which are explicitly given by

$$(\xi_i, \eta_j) = \left(-\cos\left(\frac{i\pi}{N}\right), -\cos\left(\frac{j\pi}{N}\right) \right), \quad i, j = 0, \ldots, N.$$

In the following we write the spectral derivatives. First one has to introduce the transformation matrices from physical space to coefficient space. Since we employ a Chebyshev expansion we obtain the following matrix:

$$T = (t_{i,j}) = \left(\cos\left(i\frac{j\pi}{N} \right) \right), \quad i,j = 0,\dots,N.$$

Further we need the differentation matrix in the Chebyshev coefficient space which is explicitly given by $\hat{D} = (\hat{d}_{i,j}) \in \mathbb{R}^{N+1,N+1}$ with

$$\hat{d}_{i,j} = \begin{cases} \frac{2j}{c_i} & , \quad j = i+1, i+3, \dots, N \\ 0 & , \quad \text{else} \end{cases}$$

and

$$c_i = \begin{cases} 2 & , \quad i = 0 \\ 1 & , \quad \text{else.} \end{cases}$$

Now we are able to write explicitly the spectral derivative matrix D for the first derivative which is given by

$$D = T\hat{D}T^{-1} \in \mathbb{R}^{N+1,N+1}.$$

The spectral operator can be efficiently evaluated by Fast Fourier Transformations (FFTs) in $\mathcal{O}(N \log N)$ arithmetic operations. We further introduce the identity matrix $I \in \mathbb{R}^{N+1,N+1}$. By tensor product representation $A \otimes B = (Ab_{i,j})_{i,j}$ we are now able to write the spectral derivatives:

$$\frac{\partial}{\partial x} \cong D_1 := D \otimes I \quad , \quad \frac{\partial}{\partial y} \cong D_2 := I \otimes D.$$

To decompose the domain Ω into quadratic elements $\Omega_{i,j} := (x_{i-1}, x_i) \times (y_{j-1}, y_j)$, $i,j = 1,\dots,\sqrt{K}$, where K denotes the number of elements, we define the element borders for an equidistant decomposition by:

$$x_i := -1 + i\frac{2}{\sqrt{K}}, \quad y_j := -1 + j\frac{2}{\sqrt{K}}, \quad i,j = 0,\dots,\sqrt{K}.$$

Now the collocation nodes and the differentation matrices on the k^{th} element are given by

$$x_i^k := \frac{1}{2}[(x_k - x_{k-1})\xi_i + x_{k-1} + x_k], \quad y_j^k := \frac{1}{2}[(y_k - y_{k-1})\eta_j + y_{k-1} + y_k]$$

and

$$D_{1,k} := \frac{-2}{x_k - x_{k-1}} D_1, \quad D_{2,k} := \frac{-2}{y_k - y_{k-1}} D_2$$

with $i,j = 0,\dots,N$, $k = 1,\dots,K$.
Now we are able to write the discrete spectral system on each element:

$$\begin{pmatrix} \Psi & 0 & \nu D_{2,k} & D_{1,k} \\ 0 & \Psi & -\nu D_{1,k} & D_{2,k} \\ D_{2,k} & -D_{1,k} & I \otimes I & 0 \\ D_{1,k} & D_{2,k} & 0 & 0 \end{pmatrix} \begin{pmatrix} u_{1,k}^N \\ u_{2,k}^N \\ \omega_k^N \\ p_k^N \end{pmatrix} = \begin{pmatrix} g_{1,k}^N \\ g_{2,k}^N \\ 0 \\ 0 \end{pmatrix} \text{ in } \bar{\Omega}_k^N,$$

for $k = 1, \ldots, K$. In the steady case Ψ and $\mathbf{g}$ are given by

$$\Psi = 0 \quad , \quad \mathbf{g} = \mathbf{f}$$

and in the unsteady case (since we use the second-order BDF scheme (5) for time integration) by

$$\Psi = \frac{3}{2\Delta t} \cdot I \quad , \quad \mathbf{g} = \mathbf{f} + \frac{2}{\Delta t}\mathbf{u} - \frac{1}{2\Delta t}\mathbf{u}.$$

At the interfaces between the elements, we require (as Heinrichs in [13]) continuity of both the functions and normal derivatives of u_1, u_2. For p we only require continuity and for ω we do not explicitly require interface conditions. The following plot shows the spectral element mesh for $K = 9$ elements and a polynomial degree of $N = 8$ on each element.

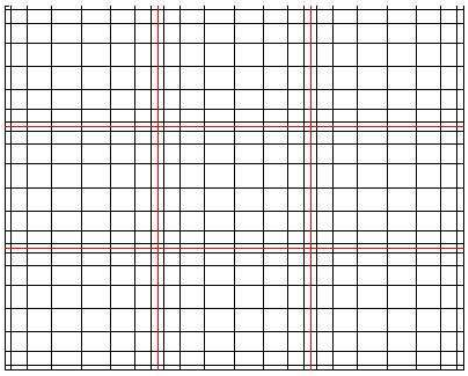


Figure 1: *Spectral element mesh for* $K = 9$ *and* $N = 8$.

Next we have to realize the discrete formulation of eqn. (4). The first way to do this is to cancel the last row and column in the matrix of the complete discrete system. As shown in subsection 4.1 (figure 2) the better way is to apply the Clenshaw-Curtis quadrature rule (see, e.g., [16]):

$$\int_\Omega p \, d\mathbf{x} \cong \sum_{i=0}^{N} \sum_{j=0}^{N} \omega_i \omega_j p(\xi_i, \eta_j)$$

where $\Omega = [-1,1]^2$ denotes the standard domain, (ξ_i, η_j) the Chebyshev Gauss-Lobatto nodes on Ω and

$$\omega_i := \begin{cases} \dfrac{1}{N^2-1} & , \quad i \in \{0, N\} \\ \dfrac{4}{N} \displaystyle\sum_{j=0}^{\frac{N}{2}} \dfrac{1}{\bar{c}_j} \dfrac{\cos\left(\frac{2\pi ij}{N}\right)}{1-4j^2} & , \quad 1 \leq i \leq N-1 \end{cases}$$

with

$$\bar{c}_j := \begin{cases} 2 & , \quad j \in \{0, N/2\} \\ 1 & , \quad 1 \leq j \leq N/2-1 \end{cases}$$

the integrations weights.
The above system of differential equations together with the boundary and interface conditions and the condition for the pressure (eqn. (4)) are written into a matrix A and compiled into an overdetermined system $Az = r$.
For the solution of the systems Heinrichs used, e.g., in [11], [12] and [13] the normal equations $A^T Az = A^T r$.
It is well known that the spectral derivative martices D have relatively large condition numbers

$$\kappa_2(D) = \frac{\max\limits_{||x||_2=1} ||Dx||_2}{\min\limits_{||x||_2=1} ||Dx||_2}$$

and the use of the corresponding normal equations lead to systems with even larger condition numbers ($\kappa_2(D^T D) \cong \kappa_2(D)^2$)). Because of the roundoff errors and the large condition numbers of the systems one cannot obtain the best quality of approximations. Here we want to avoid the normal equations to get better approximations and so we make use of a direct solver for the system $Az = r$ by using the QR decomposition (computed with MATLAB 7.3.0), see, e.g., [20], of the matrix A where we achieved the following system:

$$Az = r \Longleftrightarrow QRz = r.$$

If $A \in \mathbb{R}^{m,n}$ with $m > n$ then $Q \in \mathbb{R}^{m,m}$ is an orthogonal matrix (i.e. $Q^{-1} = Q^T$) and $R \in \mathbb{R}^{m,n}$ is an upper triangluar matrix of the type

$$R = \begin{bmatrix} \tilde{R} \\ \tilde{0} \end{bmatrix} \quad , \quad \tilde{R} \in \mathbb{R}^{n,n} \quad , \quad \tilde{0} \in \mathbb{R}^{m-n,n}.$$

Consequently, we obtain

$$\psi := Q^T r \quad , \quad Rz = \psi.$$

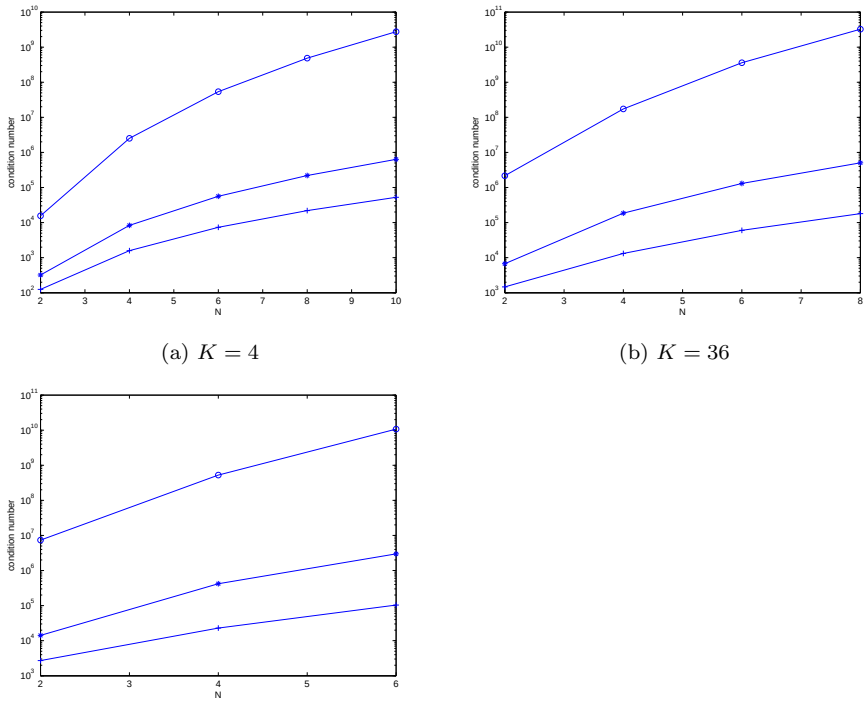
Furthermore we used the pseudoinverse, see, e.g., [20], (also known as Moore-Penrose inverse) A^+ of the matrix A to solve the system $Az = r$. The numerical experiments have shown that the approximation results by using the pseudoinverse are the same as using QR decomposition. The disadvantages of using pseudoinverses are higher computational costs (see subsection 4.1, figure 5).

4 Numerical simulations

We consider the steady and unsteady Stokes equations. In the following two subsections we denote by $\widetilde{A}$ the matrix obtained by cancelling the last column and the last row of the complete system and by $\widehat{A}$ the one obtained by including the additional Clenshaw-Curtis quadrature condition to avoid the pressure constant. The condition numbers are computed by means of a singular value decomposition of the associated matrix.

4.1 The steady Stokes equations

First, we consider the steady case of the Stokes equations. Figure 2 shows the condition numbers $\kappa_2(\widehat{A})$, $\kappa_2(\widetilde{A})$ and $\kappa_2(\widehat{A}^T\widehat{A})$ for $K = 4$, $K = 36$ and $K = 64$ elements with different polynomial degrees N. For higher element numbers and high polynomial degrees we do not list the condition numbers because of the influence of the round-off errors.
The condition numbers are rigorously reduced if we do not use the normal equations. A further reduction of the condition numbers is obtained if we use the Clenshaw-Curtis quadrature rule to avoid the pressure constant instead of cancelling one row and one column of the complete system matrix.
For other parameters K and N we obtain similar results and so we do not list them here.



(a) $K = 4$

(b) $K = 36$

(c) $K = 64$

Figure 2: *Condition numbers for* $K = 4$, $K = 36$, $K = 64$ *elements and different polynomial degrees* N $(\kappa_2(\widehat{A}) : +;\ \kappa_2(\widetilde{A}) : *;\ \kappa_2(\widehat{A}^T\widehat{A}) : o)$.

The convergence rates of the least-squares spectral collocation scheme is demonstrated by means of the model problem also introduced in [7] with $\nu = 1$. The exact velocity components and the pressure are defined on the square $\Omega := [-1, 1]^2$ by

$$
\begin{aligned}
u_1(x,y,t) &:= \cos(\gamma t)\sin\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi y}{2}\right) && (6)\\
u_2(x,y,t) &:= -\cos(\gamma t)\cos\left(\frac{\pi x}{2}\right)\sin\left(\frac{\pi y}{2}\right) && (7)\\
p(x,y,t) &:= \frac{1}{4}\cos^2(\gamma t)(\cos(\pi x)+\cos(\pi y)) + 10(x+y)\cos(\gamma t). && (8)
\end{aligned}
$$

This exact solution satisfies the Stokes equations if the following forcing term is used

$$
\begin{aligned}
\mathbf{f}(x,y) = &\begin{pmatrix} \nu\dfrac{\pi^2}{2}\cos(\gamma t)\sin\left(\dfrac{\pi x}{2}\right)\cos\left(\dfrac{\pi y}{2}\right) \\ -\nu\dfrac{\pi^2}{2}\cos(\gamma t)\cos\left(\dfrac{\pi x}{2}\right)\sin\left(\dfrac{\pi y}{2}\right) \end{pmatrix} \\
&- \begin{pmatrix} \dfrac{\pi}{4}\cos^2(\gamma t)\sin(\pi x) - 10\cos(\gamma t) \\ \dfrac{\pi}{4}\cos^2(\gamma t)\sin(\pi y) - 10\cos(\gamma t) \end{pmatrix} \\
&+ \begin{pmatrix} -\gamma\sin(\gamma t)\sin\left(\dfrac{\pi x}{2}\right)\cos\left(\dfrac{\pi y}{2}\right) \\ \gamma\sin(\gamma t)\cos\left(\dfrac{\pi x}{2}\right)\sin\left(\dfrac{\pi y}{2}\right) \end{pmatrix}.
\end{aligned} \qquad (9)
$$

For the steady case of the Stokes equations we set $\gamma = 0$.
We use the QR decomposition of the matrix $\widehat{A}$ to solve the discrete algebraic systems and numerically calculate the discrete L^2-error norms of the velocity components and the pressure. The corresponding numerical results are presented in the Tables 1 - 3.

N	$\|\|u_1 - u_1^N\|\|_{L^2}$	$\|\|u_2 - u_2^N\|\|_{L^2}$	$\|\|p - p^N\|\|_{L^2}$
$N = 2$	$1.797 \cdot 10^{-1}$	$1.809 \cdot 10^{-1}$	$6.088 \cdot 10^{-1}$
$N = 4$	$3.082 \cdot 10^{-3}$	$3.042 \cdot 10^{-3}$	$5.869 \cdot 10^{-2}$
$N = 6$	$1.340 \cdot 10^{-4}$	$1.311 \cdot 10^{-4}$	$7.061 \cdot 10^{-3}$
$N = 8$	$2.885 \cdot 10^{-6}$	$2.844 \cdot 10^{-6}$	$1.939 \cdot 10^{-4}$
$N = 10$	$1.938 \cdot 10^{-8}$	$1.936 \cdot 10^{-8}$	$2.008 \cdot 10^{-6}$
$N = 12$	$5.621 \cdot 10^{-11}$	$5.622 \cdot 10^{-11}$	$8.636 \cdot 10^{-9}$
$N = 14$	$4.425 \cdot 10^{-13}$	$4.689 \cdot 10^{-13}$	$4.852 \cdot 10^{-11}$
$N = 16$	$4.316 \cdot 10^{-13}$	$4.041 \cdot 10^{-13}$	$2.316 \cdot 10^{-11}$
$N = 18$	$5.207 \cdot 10^{-13}$	$4.977 \cdot 10^{-13}$	$8.734 \cdot 10^{-11}$
$N = 20$	$6.956 \cdot 10^{-13}$	$7.618 \cdot 10^{-13}$	$8.395 \cdot 10^{-11}$

Table 1: *L^2-errors of the velocity components and the pressure for $K = 4$ elements.*

N	$\|\|u_1 - u_1^N\|\|_{L^2}$	$\|\|u_2 - u_2^N\|\|_{L^2}$	$\|\|p - p^N\|\|_{L^2}$
$N = 2$	$4.280 \cdot 10^{-2}$	$4.434 \cdot 10^{-2}$	$4.128 \cdot 10^{-1}$
$N = 4$	$4.101 \cdot 10^{-5}$	$4.448 \cdot 10^{-5}$	$1.264 \cdot 10^{-3}$
$N = 6$	$3.285 \cdot 10^{-7}$	$3.147 \cdot 10^{-7}$	$1.082 \cdot 10^{-5}$
$N = 8$	$9.049 \cdot 10^{-10}$	$8.933 \cdot 10^{-10}$	$8.119 \cdot 10^{-8}$

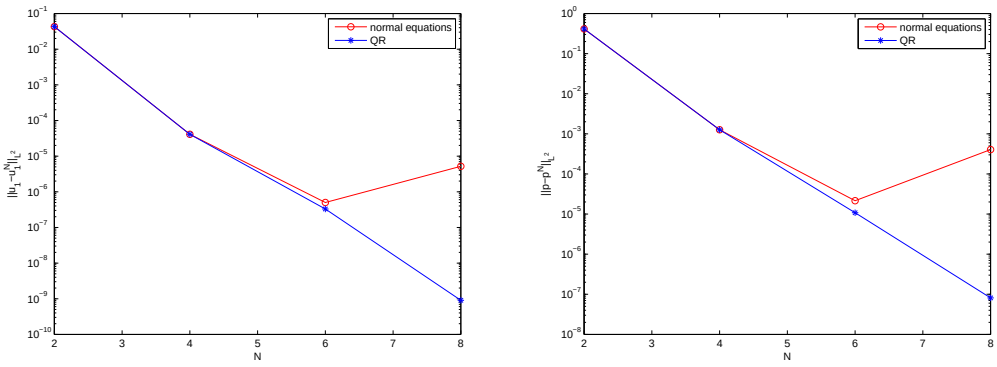
Table 2: *L^2-errors of the velocity components and the pressure for $K = 36$ elements.*

N	$\|\|u_1 - u_1^N\|\|_{L^2}$	$\|\|u_2 - u_2^N\|\|_{L^2}$	$\|\|p - p^N\|\|_{L^2}$
$N = 2$	$2.711 \cdot 10^{-2}$	$2.884 \cdot 10^{-2}$	$3.382 \cdot 10^{-1}$
$N = 4$	$1.425 \cdot 10^{-5}$	$1.562 \cdot 10^{-5}$	$5.011 \cdot 10^{-4}$
$N = 6$	$6.291 \cdot 10^{-8}$	$6.072 \cdot 10^{-8}$	$2.335 \cdot 10^{-6}$

Table 3: *L^2-errors of the velocity components and the pressure for $K = 64$ elements.*

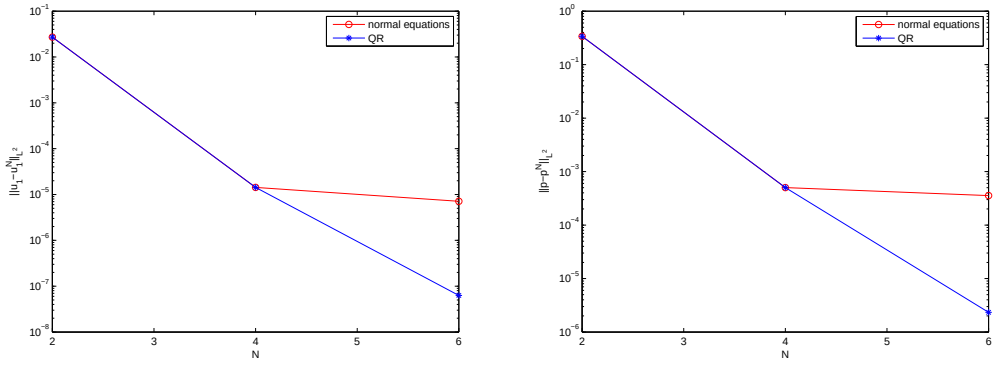
Tables 1-3 show the high spectral accuracy of our scheme if the number of elements is constant and the polynomial degree increases. If we compare the approximation errors of the same polynomial degree with different numbers of elements we observe the expected slight improvement in the results.
Table 1 obviously shows the influence of round-off errors for $N \geq 16$.

In the figures 3 and 4 we compare the approximation errors by solving the normal equations $\widehat{A}^T \widehat{A} z = \widehat{A}^T r$ and by solving the system $\widehat{A} z = r$ with QR decomposition. We obtain the same errors if the polynomial degree is low, i.e. $N = 2, 4$, but if the polynomial degree increases the errors obtained by normal equations increase if a particular N is exceeded. The reason of this behaviour is the high condition numbers of the normal equations and thus the strong influence of round-off errors. Errors obtained by QR decomposition continuous decrease by increasing N. By using QR decomposition we avoid the very high condition numbers and so we obtain the improved approximations.
Since we have an analogue performance of the velocity component u_2 we here just show the results for u_1.



(a) (b)

Figure 3: L^2-*error of* (a) *the velocity component* u_1 *and* (b) *the pressure, obtained by the normal equations and by the* QR *decomposition for* $K = 36$.



(a) (b)

Figure 4: L^2-*error of* (a) *the velocity component* u_1 *and* (b) *the pressure, obtained by the normal equations and by the* QR *decomposition for* $K = 64$.

Next we check the numerical results obtained by solving the system of equations with the pseudoinverse $\widehat{A}^+$ of the matrix $\widehat{A}$. We got the same numerical results as by using QR decomposition and so we do not show the results here.

For comparing computational costs we show in figure 5 the used CPU-times for solving the system of equations by normal equations, QR decomposition and pseudoinverse. All results are computed with MATLAB 7.3.0 with the following code:

- normal equations: `z=(Â'*Â)\(Â'*r)`
- QR decomposition: `z=Â\r`
- pseudoinverse: `z=pinv(Â)*r`

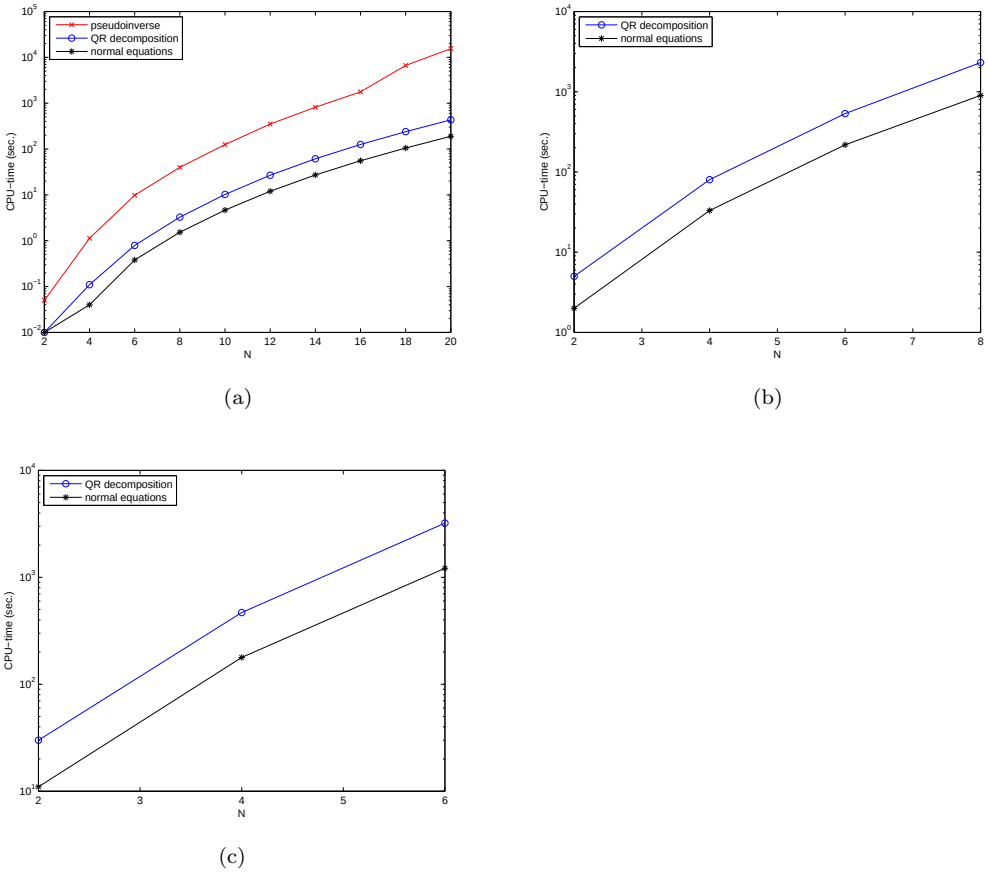


Figure 5: *CPU-time required for solving the system of linear equations with pseudoinverse* ($\times$)*, QR decomposition* (o) *and the normal equations* ($*$) *on* (a) 4, (b) 36 *and* (c) 64 *elements.*

Since the use of the pseudoinverse needs extremely more time and produces same accuracy as the use of QR decomposition we only used QR decomposition for solving the systems. The application of normal equations is less expensive but the quality of approximation is worse than the one obtained by using QR decomposition. In the following section we apply our scheme to the unsteady Stokes equations.

4.2 The unsteady Stokes equations

Now we consider the unsteady case of the Stokes equations. The exact velocity components and the pressure are defined as in (6)-(8), the corresponding forcing term by (9) and we recall that $\nu = \frac{1}{Re}$ where Re denotes the Reynolds number. Because we consider the unsteady case we set $\gamma = 5$ as in [7]. From [11] it is known that for a well balanced system it is recommended to scale the momentum equations by Δt. By numerical experiments we observed the same. Without scaling the incompressibility condition is no more fulfilled after time integration. Figure 6 shows $||\nabla \cdot \mathbf{u}||$ for $K = 4$, $N = 10$, $Re = 1$ and $\Delta t = \frac{1}{58}$ in case without

scaling.

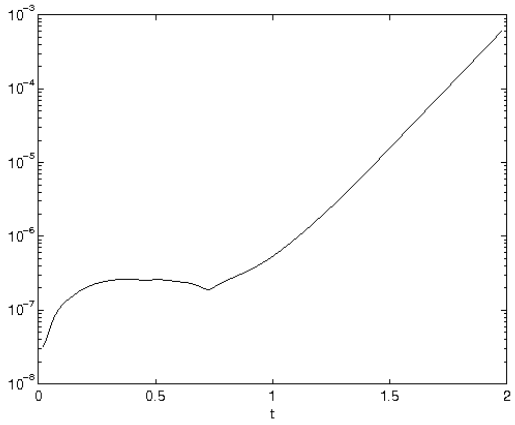
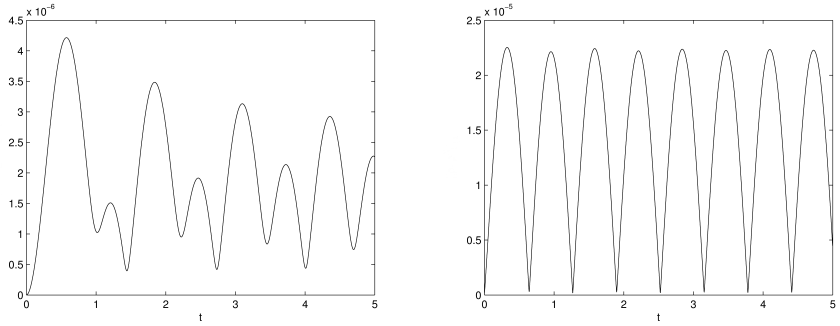


Figure 6: *Temporal evolution of* $||\nabla \cdot \mathbf{u}||$ *for* $K = 4$, $N = 10$, $Re = 1$, $\Delta t = \frac{1}{58}$ *in the case without scaling.*

In the scaled case we obtained a stable scheme. Figure 7 shows temporal evolution of the L^2-errors in the velocity components and in the pressure. We observe no enlargement of the oscillating errors in time, expressing stability of the numerical solution.



(a) (b)

Figure 7: *Temporal evolution of* (a) *the velocity error and* (b) *the pressure error for* $Re = 100$, $\Delta t = \frac{1}{1000}$, $K = 4$ *and* $N = 10$.

Since the numerical results in the unsteady case are similar to those in the steady case we here only show a few results. In Table 4 we demonstrate the condition numbers of the matrices $\widehat{A}$ and recall that $\kappa_2(\widehat{A}^T\widehat{A}) \cong \kappa_2(\widehat{A})^2$.

Re	Δt	$\kappa_2(\widehat{A})$
100	$\frac{1}{10}$	$3.1067 \cdot 10^4$
100	$\frac{1}{100}$	$9.0105 \cdot 10^4$
1000	$\frac{1}{100}$	$8.7253 \cdot 10^4$

Table 4: *Condition numbers of the scaled matrix* $\widehat{A}$ *for* $K = 4$, $N = 10$, *different Reynolds numbers and different timesteps.*

In the next three tables we show the approximation errors for the unsteady case of the Stokes equations. The time derivative is approximated by the second order BDF scheme (5). We set

$$\begin{aligned} E_{u_1} &:= \max\{||u_1 - u_1^N||_{L^2} \quad : \quad t \in [0,1]\}, \\ E_{u_2} &:= \max\{||u_2 - u_2^N||_{L^2} \quad : \quad t \in [0,1]\}, \\ E_p &:= \max\{||p - p^N||_{L^2} \quad : \quad t \in [0,1]\} \end{aligned}$$

since figure 7 shows that the maximum error is obtained in $[0, 1]$.
Tables 5-7 show the good performance of the here presented scheme for time-depend Stokes problems.

Δt	$E_{u_1} = E_{u_2}$	Ratio	E_p	Ratio
$\frac{1}{10}$	$2.885 \cdot 10^{-3}$	—	$2.496 \cdot 10^{-1}$	—
$\frac{1}{20}$	$7.386 \cdot 10^{-4}$	3.906	$6.331 \cdot 10^{-2}$	3.943
$\frac{1}{40}$	$1.858 \cdot 10^{-4}$	3.975	$1.586 \cdot 10^{-2}$	3.992
$\frac{1}{80}$	$4.657 \cdot 10^{-5}$	3.990	$3.973 \cdot 10^{-3}$	3.992

Table 5: *Approximation errors for* $K = 4$, $N = 10$ *and* $Re = 1$.

Δt	$E_{u_1} = E_{u_2}$	Ratio	E_p	Ratio
$\frac{1}{10}$	$1.326 \cdot 10^{-1}$	—	$5.354 \cdot 10^{-1}$	—
$\frac{1}{20}$	$3.563 \cdot 10^{-2}$	3.722	$1.358 \cdot 10^{-1}$	3.943
$\frac{1}{40}$	$9.085 \cdot 10^{-3}$	3.922	$3.404 \cdot 10^{-2}$	3.989
$\frac{1}{80}$	$2.281 \cdot 10^{-3}$	3.983	$8.532 \cdot 10^{-3}$	3.990

Table 6: *Approximation errors for* $K = 36$, $N = 8$ *and* $Re = 100$.

Δt	$E_{u_1} = E_{u_2}$	Ratio	E_p	Ratio
$\frac{1}{10}$	$1.890 \cdot 10^{-1}$	—	$7.322 \cdot 10^{-1}$	—
$\frac{1}{20}$	$5.067 \cdot 10^{-2}$	3.730	$1.855 \cdot 10^{-1}$	3.947
$\frac{1}{40}$	$1.292 \cdot 10^{-2}$	3.922	$4.656 \cdot 10^{-2}$	3.984
$\frac{1}{80}$	$3.244 \cdot 10^{-3}$	3.983	$1.166 \cdot 10^{-2}$	3.993

Table 7: *Approximation errors for* $K = 64$, $N = 6$ *and* $Re = 100$.

5 Conclusion

We presented a least-squares spectral collocation scheme for the steady and unsteady Stokes equations where the original domain has been decomposed into quadrilateral subelements. To avoid high condition numbers of normal equations a direct solver (QR decomposition of the matrices) was used for the overdetermined systems. The numerical simulations confirm the high accuracy of the proposed spectral least-squares scheme and solving the overdetermined systems with QR decomposition yields better approximation results. The computational cost of QR decomposition is higher than using normal equations. Using pseudoinverse shows same results as using QR decomposition but causes much higher computational costs.
In the unsteady case we have shown the good performance of the here presented scheme for different Reynolds numbers, different time steps and various number of elements.
In the second part of the paper we intend to present results for the incompressible Navier-Stokes equations. First results are already contained in [14].

References

[1] C. Bernardi C. Canuto, Y. Maday. *Generalized inf-sup condition for Chebyshev approximations to Navier-Stokes equations.* C.R. Acad. Sci. Paris, 303(serie I):971–974,1986. Springer, 1988.

[2] C. Canuto, M.Y.Hussaini, A. Quarteroni, T.A. Zang. *Spectral Methods in Fluid Dynamics.* Springer Series in Computational Physics. Springer, 1988.

[3] M. O. Deville, P. F. Fischer, E. H. Mund. *High-Order Methods for Incompressible Fluid Flow.* Cambridge Monographs on Applied and Computational Mathematics. Cambridge University Press, New York, 2002.

[4] H. Eisen, W. Heinrichs. *A new method of stabilization for singular perturbation problems.* SIAM J. Numer. Anal., 29:107–122, 1992.

[5] D. Gottlieb, S. A. Orszag. *Numerical Analysis of Spectral Methods: Theory and Applications.* CBMS-NSF Regional Conference Series in Applied Mathematics, no.26. SIAM, 1977.

[6] M. I. Gerritsma, M. M. J. Proot. *Analysis of a discontinuous least-squares spectral element method.* J. Sci. Comput., 17:297–306, 2002.

[7] H. Haschke and W. Heinrichs. *Splitting Techniques with Staggered Grids for the Navier-Stokes Equations in the 2D Case.* J. Comput. Phys., 168(1):131–154, 2001.

[8] W. Heinrichs. *Nonconforming Spectral Collocation.* `www.uni-essen.de/ingmath/people/heinrichs/mortar.ps`.

[9] W. Heinrichs. *Splitting techniques for the pseudospectral approximation of the unsteady Stokes equations.* SIAM J. Numer. Anal., 30(1):19–39, 1993.

[10] W. Heinrichs. *Least-Squares Spectral Collocation for Discontinuous and Singular Perturbation Problems.* J. Comput. Appl. Math., 157(2):329–345, 2003.

[11] W. Heinrichs. *Least-Squares Spectral Collocation for the Navier-Stokes Equations.* J. Sci. Comput., 21(1):81–90, 2004.

[12] W. Heinrichs. *Least-squares spectral collocation with the overlapping Schwarz method for the incompressible Navier-Stokes equations.* Numer. Algorithms., (43):61–73, 2006.

[13] W. Heinrichs. *An adaptive least-squares spectral collocation method with triangular elements for the incompressible Navier-Stokes equations.* J. Eng. Math., 56(3):337–350, 2006.

[14] T. Kattelans. *Spektrale Least-Squares Verfahren für inkompressible Navier-Stokes-Gleichungen.* Diploma thesis (in German), University of Duisburg-Essen, 2007.

[15] S. A. Orszag. *Spectral methods for problems in complex geometries.* J. Comput. Phys., 37:70–92, 1980.

[16] R. Peyret. *Spectral Methods for Incompressible Viscous Flow.* Springer, 2002.

[17] M. M. J. Proot and M. I. Gerritsma. *A Least-Squares Spectral Element Formulation for the Stokes Problem.* J. Sci. Comput., 17(1-4):285–296, 2002.

[18] M. M. J. Proot and M. I. Gerritsma. *Least-Squares Spectral Elements Applied to the Stokes Problem.* J. Comput. Phys., 181(2):454–477, 2002.

[19] M. M. J. Proot and M. I. Gerritsma. *Application of the least-squares spectral element method using Chebyshev polynomials to solve the incompressible Navier-Stokes equations.* Numer. Algorithms., 38:155–172, 2005.

[20] A. Quarteroni and A. Valli. *Numerical Approximation of Partial Differential Equations.* Springer, 1994.

[21] E. Rønquist. *Optimal spectral element methods for the unsteady three dimensional incompressible Navier-Stokes equations.* Phd. thesis, Massachusetts Institute of Technology, Cambridge, Massachusetts, 1988.