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Variational source condition for ill-posed backward nonlinear Maxwell's equations

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Abstract

This paper examines the mathematical analysis of an electromagnetic inverse problem governed by nonlinear evolutionary Maxwell's equations. The aim of the inverse problem is to recover electromagnetic fields at the past time by noisy measurement data at the present time. We consider the Tikhonov regularization method to cope with the ill-posedness of the governing backward nonlinear Maxwell's equations. By means of the semigroup theory, we study its convergence analysis and derive optimality conditions through a rigorous first-order analysis and adjoint calculus. The final part of the paper is focused on the convergence rate analysis of the Tikhonov regularization method under a variational source condition (VSC), which leads to power-type convergence rates. Employing the spectral theory, the complex interpolation theory and fractional Sobolev spaces, we validate the proposed VSC on account of an appropriate regularity assumption on the exact initial data and the material parameters.

Keywords: nonlinear Maxwell's equations, ill-posed backward evolution equations, Tikhonov regularization, variational source condition, inverse problems

1. Introduction

This paper is concerned with the mathematical analysis for an electromagnetic inverse problem governed by Maxwell's equations. The inverse problem is to recover (unknown) electromagnetic fields at the past time ($t = 0$) by measurement at the present time ($t = T > 0$). As the governing forward problem, we focus on nonlinear hyperbolic Maxwell's equations, where the nonlinearity arises from the modelling of nonlinear material properties as encountered in

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various electromagnetic materials such as ferromagnetic materials or superconductors (see [24, 39, 41]). To be more precise, we consider the following nonlinear hyperbolic Maxwell system:

$$\begin{cases} \varepsilon\partial_t \mathbf{E}(x,t) - \nabla\times\mathbf{H} = \mathbf{F}_1(t,x,\mathbf{E}(x,t),\mathbf{H}(x,t)) & \text{in } \Omega\times(0,T), \\ \mu\partial_t \mathbf{H}(x,t) + \nabla\times\mathbf{E} = \mathbf{F}_2(t,x,\mathbf{E}(x,t),\mathbf{H}(x,t)) & \text{in } \Omega\times(0,T), \\ \qquad \mathbf{E}(x,t)\times\mathbf{n} = 0 & \text{on } \partial\Omega\times(0,T). \end{cases} \tag{1.1}$$

In the setting of (1.1), $\Omega\subset\mathbb{R}^3$ denotes a bounded Lipschitz domain and $\mathbf{F}_1,\mathbf{F}_2:[0,T]\times\Omega\times\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}^3$ are (given) nonlinear functions. Furthermore, $\mathbf{E}:\Omega\times[0,T]\to\mathbb{R}^3$ denotes the electric field, $\mathbf{H}:\Omega\times[0,T]\to\mathbb{R}^3$ the magnetic field, $\varepsilon:\Omega\to\mathbb{R}^{3\times3}$ the electric permittivity and $\mu:\Omega\to\mathbb{R}^{3\times3}$ the magnetic permeability. The precise mathematical assumptions for all the data involved in (1.1) will be specified in section 2. Let us underline that, in the Maxwell forward system (1.1), the material parameters ϵ and μ are assumed to be known data. We refer to [11, 26] for ill-posed identification problems of the electric permittivity and the magnetic permeability in the linear counterpart to (1.1).

Given data for the electromagnetic fields at the final time $(\mathbf{e}^\dagger,\mathbf{h}^\dagger)$, our goal is to recover the initial value $(\mathbf{E}(\cdot,0),\mathbf{H}(\cdot,0)) =: (\mathbf{u}^\dagger,\mathbf{v}^\dagger)$ in the space

$$\mathbf{Y} := \{(\mathbf{u},\mathbf{v})\in\mathbf{H_0}(\mathbf{curl})\times\mathbf{H}(\mathbf{curl}) \mid \varepsilon\mathbf{u}\in\mathbf{H}(\mathrm{div}),\ \ \mu\mathbf{v}\in\mathbf{H_0}(\mathrm{div})\},$$

where the imposed divergence conditions in $\mathbf{Y}$ are motivated by the physical Gauss law for magnetic and electric fields. Note that, since ϵ and μ are only of class $L^\infty(\Omega)^{3\times3}$, every element of $\mathbf{Y}$ does not necessarily enjoy a higher regularity in $\mathbf{H}^s(\Omega)\times\mathbf{H}^s(\Omega)$ for $s>0$. However, by the Maxwell compactness embedding theory [29, 34], the embedding $\mathbf{Y}\hookrightarrow\mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Omega)$ is compact. Our analysis will benefit from this compactness result. The considered inverse problem is ill-posed in the following sense: If we replace the exact terminal data $(\mathbf{e}^\dagger,\mathbf{h}^\dagger)$ by a noisy pattern $(\mathbf{e}^\delta,\mathbf{h}^\delta)$ satisfying

$$\|(\mathbf{e}^\delta,\mathbf{h}^\delta)-(\mathbf{e}^\dagger,\mathbf{h}^\dagger)\|_{\mathbf{L}^2_\epsilon(\Omega)\times\mathbf{L}^2_\mu(\Omega)}\leqslant\delta,$$

where the parameter $\delta>0$ is used to represent a noise level in the data, then the initial value of the (mild) solution to (1.1) may not belong to the space $\mathbf{Y}$. Even if it belongs to $\mathbf{Y}$ and δ is small, the solution could still be far from the exact initial value $(\mathbf{u}^\dagger,\mathbf{v}^\dagger)\in\mathbf{Y}$.

To deal with the ill-posedness, we consider the Tikhonov regularization technique by solving a least-squares nonlinear minimization problem. Our main goal is to examine the convergence rate of the regularized solution under an appropriate choice of the noise level δ and the Tikhonov regularization parameter. To obtain the convergence rate, one usually requires an additional smoothness assumption on the true solution, well-known as the so-called *source condition*. In general, (classical) source conditions could be restrictive since they require the Fréchet differentiability of the forward operator and further properties on the adjoint of the Fréchet-derivative (see [10, 12, 23, 25]). Our present work shall focus on the so-called *variational source condition* (VSC). The concept of VSC was originally introduced by Hofmann *et al* [18] for the case of a linear index function Ψ. Convergence rates for a general index function Ψ were shown independently in [6, 13, 16]. We also refer to [14] for a modified proof of the convergence rate result by [16]. In contrast to the classical source condition, VSC does not require any differentiability assumption on the forward operator. More importantly, under an appropriate parameter choice rule (see [22]), convergence rates can be deduced from VSC in a straightforward manner.

In literature, there is only a small number of contributions towards the verification of VSC for inverse problems governed by partial differential equations. For abstract linear operators and ℓ^p penalties with respect to certain bases, we refer to [2, 5, 7] and references therein. Recently, Hohage and Weidling [19, 20] shown the validity of VSC for the Tikhonov regularization of inverse scattering problems. In particular, based on VSC, they obtained convergence with logarithmic-type rates for the corresponding regularized solution. For more details between VSC and classical source conditions, we refer the reader to [19–21]. See also [7] concerning recent results on VSC for elastic-net regularizations.

In this work, we propose VSC for the Tikhonov regularization of the ill-posed backward nonlinear Maxwell's equation (1.1). The main novelty of our contribution includes the verification of VSC by means of the spectral theory and the complex interpolation theory under an appropriate regularity assumption on $(\mathbf{e}^\dagger, \mathbf{h}^\dagger)$ and a piecewise regularity assumption on ε and μ (see **(A4)**, p 17). In particular, this assumption is related to the physical material structure of the medium Ω consisting of different heterogeneous materials. Since our techniques are different from those proposed in [19–21], we believe that our results may help enrich the works on VSC for inverse problems governed by partial differential equations. In addition to the verification of VSC, we also examine the sensitivity analysis of the associated Maxwell forward operator and establish its Gâteaux-differentiability property. In particular, this result allows us to develop adjoint techniques and derive first-order optimality conditions for the associated Tikhonov regularization problem. Then, based on VSC, we obtain convergence rates for the corresponding adjoint state. To our best knowledge, these results have not been obtained in the literature of PDE-constrained optimization (see [38–40]).

The outline of the paper is as follows. In section 2, we provide the mathematical formulation for the ill-posed backward nonlinear Maxwell's equation (1.1) and the associated Tikhohov regularization. Section 3 is concerned with the well-posedneess of Tikhohov regularization and its first-order sensitivity analysis. In section 4, we establish the validity of VSC for the considered inverse problem and derive convergence rates for regularised solutions and adjoint states.

2. Preliminaries and mathematical formulation

Throughout this paper, for a given Hilbert space H, we denote by $(\cdot,\cdot)_H$ the standard inner product and by $\|\cdot\|_H$ the standard norm of H. If H is continuously embedded into a normed vector space V, then we write $V \hookrightarrow H$. The notation $\mathcal{L}(X,Y)$ stands for the space of all bounded linear operator from a normed space X into another normed space Y endowed with the standard operator norm $\|\cdot\|_{\mathcal{L}(X,Y)}$. If $Y = X$, then we use the abbreviation $\mathcal{L}(X)$. For an interval $J \subset \mathbb{R}$, $1 \leqslant p \leqslant \infty$, and a normed space X, let $L^p(J;X)$ denote the classical L^p-Bochner space. Moreover, $C([a,b];X)$ denotes the Banach space of all continuous function from $[a,b]$ to X. A bold typeface is used to indicate a three-dimensional vector function or a Hilbert space of three-dimensional vector functions. In our analysis, we mainly deal with the following Hilbert spaces:

$$\begin{aligned}
\mathbf{H}(\mathbf{curl}) &:= \{\mathbf{q} \in \mathbf{L}^2(\Omega) \mid \mathbf{curl}\,\mathbf{q} \in \mathbf{L}^2(\Omega)\},\\
\mathbf{H_0}(\mathbf{curl}) &:= \{\mathbf{q} \in \mathbf{H}(\mathbf{curl}) \mid \mathbf{q} \times \mathbf{n} = 0 \text{ on } \partial\Omega\},\\
\mathbf{H}(\mathrm{div}) &:= \{\mathbf{q} \in \mathbf{L}^2(\Omega) \mid \mathrm{div}\,\mathbf{q} \in L^2(\Omega)\},\\
\mathbf{H_0}(\mathrm{div}) &:= \{\mathbf{q} \in \mathbf{H}(\mathrm{div}) \mid \mathbf{q} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\},
\end{aligned}$$

where the **curl**- and div -operators as well as the normal and tangential traces are understood in the sense of distributions (see [15]). In this paper, we make use of the notation $A \lesssim B$ to indicate that $A \leqslant CB$ for some positive constant C that is independent of A and B.

For a symmetric and uniformly positive definite function $\alpha \in L^\infty(\Omega)^{3\times 3}$, the notation $\mathbf{L}^2_\alpha(\Omega)$ stands for the α-weighted $\mathbf{L^2}(\Omega)$-space with the weighted scalar product $(\alpha\cdot,\cdot)_{\mathbf{L^2}(\Omega)}$. Then, we define the weighted Hilbert space $\mathbf{X} := \mathbf{L}^2_\varepsilon(\Omega) \times \mathbf{L}^2_\mu(\Omega)$, equipped with the weighted scalar product

$$((\mathbf{u},\mathbf{v}),(\hat{\mathbf{u}},\hat{\mathbf{v}}))_{\mathbf{X}} := (\varepsilon\mathbf{u},\hat{\mathbf{u}})_{\mathbf{L^2}(\Omega)} + (\mu\mathbf{v},\hat{\mathbf{v}})_{\mathbf{L^2}(\Omega)} \quad \forall\ (\mathbf{u},\mathbf{v}),\ (\hat{\mathbf{u}},\hat{\mathbf{v}}) \in \mathbf{X}.$$

Let us now introduce the (unbounded) Maxwell operator

$$\mathcal{A} : D(\mathcal{A}) \subset \mathbf{X} \to \mathbf{X}, \quad \mathcal{A} := -\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & -\mathbf{curl} \\ \mathbf{curl} & 0 \end{pmatrix},$$

whose domain is given by $D(\mathcal{A}) := \mathbf{H_0}(\mathbf{curl}) \times \mathbf{H}(\mathbf{curl})$.

Throughout this paper, we assume that the following standing assumptions hold:

(A0) Let the electric permittivity and the magetic permeability $\varepsilon, \mu : \Omega \to \mathbb{R}^{3\times 3}$ be of class $L^\infty(\Omega)^{3\times 3}$, symmetric and uniformly positive definite in the sense that there exist positive real numbers $\underline{\epsilon}$ and $\underline{\mu}$ such that

$$\xi^T\varepsilon(x)\xi \geqslant \underline{\epsilon}|\xi|^2 \quad \text{and} \quad \xi^T\mu(x)\xi \geqslant \underline{\mu}|\xi|^2 \quad \text{for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^3.$$

(A1) For every $t \in [0,T]$, the operator $\mathbf{F}(t,\cdot) : \mathbf{X} \to \mathbf{X}$ given by

$$(\mathbf{u},\mathbf{v}) \mapsto (\mu^{-1}\mathbf{F}_1(t,x,(\mathbf{u},\mathbf{v})(x))), \varepsilon^{-1}\mathbf{F}_2(t,x,(\mathbf{u},\mathbf{v})(x))$$

is well-defined, and the mapping $\mathbf{F} : [0,T] \times \mathbf{X} \to \mathbf{X}$ is globally Lipschitz-continuous with the Lipschitz constant $L > 0$, i.e.

$$\|\mathbf{F}(t_1,(\mathbf{u}_1,\mathbf{v}_1)) - \mathbf{F}(t_2,(\mathbf{u}_2,\mathbf{v}_2))\|_{\mathbf{X}} \leqslant L(|t_1 - t_2| + \|(\mathbf{u}_1,\mathbf{v}_1) - (\mathbf{u}_2,\mathbf{v}_2)\|_{\mathbf{X}})$$

for all $t_1, t_2 \in [0,T]$ and $(\mathbf{u}_1,\mathbf{v}_1), (\mathbf{u}_2,\mathbf{v}_2) \in \mathbf{X}$.

Applying the Maxwell operator $\mathcal{A}$, the nonlinear hyperbolic Maxwell system (1.1) can be reformulate as the following abstract Cauchy problem:

$$\begin{cases} \left(\frac{\mathrm{d}}{\mathrm{d}t} - \mathcal{A}\right)(\mathbf{E},\mathbf{H})(t) = \mathbf{F}(t,\mathbf{E}(t),\mathbf{H}(t)) \quad \text{in } (0,T], \\ (\mathbf{E},\mathbf{H})(0) = (\mathbf{u},\mathbf{v}). \end{cases} \tag{2.1}$$

Obviously, $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{X} \to \mathbf{X}$ is a densely defined, and closed operator. Moreover, due to the choice of the weighted Hilbert space $\mathbf{X}$, the operator $\mathcal{A}$ is skew-adjoint, i.e. $\mathcal{A}^* = -\mathcal{A}$ with $D(\mathcal{A}^*) = D(\mathcal{A})$ (see, e.g. [4, lemma A.2 and (3.4)] for this standard result). Thus, by virtue of Stone's theorem [33, theorem 10.8, p 41], $\mathcal{A}$ generates a strongly continuous group $\{\mathbb{T}\}_{t\in\mathbb{R}}$ of unitary operators on $\mathbf{X}$.

Definition 2.1. Let $(\mathbf{u},\mathbf{v}) \in \mathbf{X}$. A continuous function $(\mathbf{E},\mathbf{H}) \in C([0,T];\mathbf{X})$ is called a mild solution of (2.1) associated with $(\mathbf{u},\mathbf{v})$ if and only if

$$(\mathbf{E},\mathbf{H})(t) = \mathbb{T}_t(\mathbf{u},\mathbf{v}) + \int_0^t \mathbb{T}_{t-s}\mathbf{F}(s,\mathbf{E}(s),\mathbf{H}(s))\mathrm{d}s \quad \forall\, t \in [0,T].$$

Thanks to the Lipschitz property **(A1)**, a classical result [33] implies that the Cauchy problem (2.1) admits for every $(\mathbf{u},\mathbf{v}) \in \mathbf{X}$ a unique mild solution $(\mathbf{E},\mathbf{H}) \in C([0,T];\mathbf{X})$. We denote the mild solution operator associated with (2.1) by

$$G : \mathbf{X} \to C([0,T];\mathbf{X}), \quad (\mathbf{u},\mathbf{v}) \mapsto (\mathbf{E},\mathbf{H}),$$

which assigns to every initial data $(\mathbf{u},\mathbf{v}) \in \mathbf{X}$ the unique mild solution $(\mathbf{E},\mathbf{H}) \in C([0,T];\mathbf{X})$ of (2.1). Furthermore, we introduce the operator

$$S : \mathbf{X} \to \mathbf{X}, \quad S(\mathbf{u},\mathbf{v}) := G(\mathbf{u},\mathbf{v})(T).$$

Let us now state the regularity assumption on the initial data we consider throughout this paper:

(**A2**) Suppose that the exact true initial value satisfies $(\mathbf{u}^\dagger,\mathbf{v}^\dagger) \in \mathbf{Y}$.

In all what follows, we set

$$(\mathbf{e}^\dagger,\mathbf{h}^\dagger) := S(\mathbf{u}^\dagger,\mathbf{v}^\dagger).$$

As pointed out in the introduction, we aim at recovering the initial value of the mild solution for (2.1) from a given measurement at final time with noise data, which we denote by $(\mathbf{e}^\delta,\mathbf{h}^\delta) \in \mathbf{X}$ satisfying

$$\|(\mathbf{e}^\delta,\mathbf{h}^\delta) - (\mathbf{e}^\dagger,\mathbf{h}^\dagger)\|_{\mathbf{X}} \leqslant \delta, \tag{2.2}$$

with $\delta > 0$ representing the noise level in the data. Then, the inverse problem reads as follows: Find $(\mathbf{u},\mathbf{v}) \in \mathbf{Y}$ such that $S(\mathbf{u},\mathbf{v}) = (\mathbf{e}^\delta,\mathbf{h}^\delta)$. Our goal is to investigate the corresponding Tikhonov regularization method:

$$\mathcal{J}_\delta^\kappa(\mathbf{u},\mathbf{v}) := \frac{1}{2}\|S(\mathbf{u},\mathbf{v}) - (\mathbf{e}^\delta,\mathbf{h}^\delta)\|_{\mathbf{X}}^2 + \frac{\kappa}{2}\|(\mathbf{u},\mathbf{v})\|_{\mathbf{Y}}^2 \to \min, \quad \text{subject to} (\mathbf{u},\mathbf{v}) \in \mathbf{Y}. \tag{2.3}$$

In the Tikhonov minimization problem (2.3), we employ the full norm of $\mathbf{Y}$, which is defined by

$$\|(\mathbf{u},\mathbf{v})\|_{\mathbf{Y}}^2 := \|\mathbf{u}\|_{\mathbf{H}(\mathbf{curl})}^2 + \|\mathbf{v}\|_{\mathbf{H}(\mathbf{curl})}^2 + \|\mathrm{div}\,(\varepsilon\mathbf{u})\|_{L^2(\Omega)}^2 + \|\mathrm{div}\,(\mu\mathbf{v})\|_{L^2(\Omega)}^2 \quad \forall(\mathbf{u},\mathbf{v}) \in \mathbf{Y}.$$

The use of the full norm is mainly required for the analysis of VSC (see section 4). If we are only interested in the existence of a minimizer for (2.3), then, thanks to (3.8), we may also replace the full norm $\|\cdot\|_{\mathbf{Y}}$ by the semi-norm

$$\begin{aligned} |(\mathbf{u},\mathbf{v})|_{\mathbf{Y}}^2 := &\|\mathbf{curl}\,\mathbf{u}\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathrm{div}\,(\varepsilon\mathbf{u})\|_{L^2(\Omega)}^2 \\ &+ \|\mathbf{curl}\,\mathbf{v}\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathrm{div}\,(\mu\mathbf{v})\|_{L^2(\Omega)}^2\ \forall(\mathbf{u},\mathbf{v}) \in \mathbf{Y}. \end{aligned} \tag{2.4}$$

Note that the Friedrichs–Poincaré-type inequalities:

$$\|\mathbf{u}\|_{\mathbf{L}^2(\Omega)} \lesssim \|\mathbf{curl}\,\mathbf{u}\|_{\mathbf{L}^2(\Omega)} + \|\mathrm{div}\,(\varepsilon\mathbf{u})\|_{L^2(\Omega)} \quad \forall\,\mathbf{u} \in \mathbf{H_0}(\mathbf{curl}) \cap \varepsilon^{-1}\mathbf{H}(\mathrm{div})$$

and

$$\|\mathbf{v}\|_{\mathbf{L}^2(\Omega)} \lesssim \|\mathbf{curl}\,\mathbf{v}\|_{\mathbf{L}^2(\Omega)} + \|\mathrm{div}\,(\mu\mathbf{v})\|_{L^2(\Omega)} \quad \forall\,\mathbf{v} \in \mathbf{H}(\mathbf{curl}) \cap \mu^{-1}\mathbf{H_0}(\mathrm{div})$$

hold true, if Ω is simply connected and its boundary is connected. This is a well-known consequence of the Maxwell compactness embedding theory [29, 34]. Since we do not make any additional geometrical assumption on Ω, the semi-norm (2.4) is not equivalent to the full norm of $\mathbf{Y}$.

We close this section by providing a classical result on the energy balance equality for every strongly continuous group of unitary operators on $\mathbf{X}$. For a proof, we refer to [42, lemma 2.1], which is based on a density argument and the semigroup theory.

Lemma 2.2. *Let* $\{\mathbb{S}_t\}_{t\in\mathbb{R}}$ *be a strongly continuous group of unitary operators on* $\mathbf{X}$*. Furthermore, suppose that* $(\mathbf{e},\mathbf{h}) \in C([0,T];\mathbf{X})$*,* $(\mathbf{e}_0,\mathbf{h_0}) \in \mathbf{X}$ *and* $(\mathbf{f_1},\mathbf{f_2}) \in L^1((0,T);\mathbf{X})$ *satisfy*

$$(\mathbf{e},\mathbf{h})(t) = \mathbb{S}_t(\mathbf{e}_0,\mathbf{h_0}) + \int_0^t \mathbb{S}_{t-s}(\mathbf{f_1},\mathbf{f_2})(s)\mathrm{d}s \quad \forall\, t \in [0,T].$$

Then, the energy balance equality

$$\|(\mathbf{e},\mathbf{h})(t)\|_{\mathbf{X}}^2 = \|(\mathbf{e}_0,\mathbf{h_0})\|_{\mathbf{X}}^2 + 2\int_0^t ((\mathbf{f_1},\mathbf{f_2})(s),(\mathbf{e},\mathbf{h})(s))_{\mathbf{X}}\mathrm{d}s \tag{2.5}$$

holds for all $t \in [0,T]$.

As investigated in [40], the energy balance equality is an important tool for the mathematical analysis of the optimal control of nonlinear evolutionary Maxwell's equations. In our case, the energy balance equality is important for the case, where $\mathbf{F}$ is monotone (see proposition 3.3).

3. Tikhonov regularization and its sensitivity analysis

In section 3.1, we first recall some results from the Tikhonov regularization theory (see [3, 14, 16, 18]). Then, in section 3.2, we derive some important properties for the Tikhonov regularization (2.3), leading to an existence and convergence result. The final part of this section is devoted to the analysis of the adjoint state associated with the Tikhonov regularization (2.3) and its convergence behavior.

3.1. Tikhonov regularization

Let W and Z be Hilbert spaces and $F : W \to Z$ be an operator with an unbounded inverse F^{-1}. Given $z \in Z$, we look for a solution $w \in W$ of the following operator equation:

$$F(w) = z. \tag{3.1}$$

This operator equation is ill-posed in the sense that a solution possibly does not exist, if the exact data z comes with small noisy, namely only the noise data z^δ of z available satisfying

$$\|z - z^\delta\|_Z \leqslant \delta,$$

for some small noisy level $\delta > 0$. Even if a solution exists, then it could be crucially far away from the original one. To obtain a stable approximation of a norm-minimizing solution of (3.1), i.e. a solution $w^\dagger \in W$ to (3.1) such that

$$F(w^\dagger) = z \quad \text{and} \quad \|w^\dagger\|_W = \min\{\|w\|_W \mid F(w) = z\}. \tag{3.2}$$

Tikhonov proposed the use of a regularized solution w_κ^δ, given by a minimizer of the following minimization problem:

$$T_\kappa^\delta(w) := \frac{1}{2}\|F(w) - z^\delta\|_Z^2 + \kappa\,\|w\|_W^2 \to \min, \quad \text{subject to} \quad w \in W. \tag{3.3}$$

We refer to [8, 31], [18, section 4.1] and [32, section 4.1] for the convergence analysis of w_κ^δ as $\kappa, \delta \to 0$.

In general, however, the convergence rate of w_κ^δ may be arbitrary slow (see [8]). To achieve an explicit convergence rate of the regularized solution w_κ^δ, one needs to choose an appropriate parameter $\kappa = \kappa(\delta, z^\delta)$ and impose an additional condition on $w^\dagger$. In the literature, this condition on $w^\dagger$ is called a *source condition*. In particular, the *variational source condition* of the form

$$\frac{\beta}{2}\|w^\dagger - w\|_W^2 \leqslant \frac{1}{2}\|w\|_W^2 - \frac{1}{2}\|w^\dagger\|_W^2 + \Psi(\|F(w^\dagger) - F(w)\|_Z) \quad \forall w \in W \tag{3.4}$$

became more popular for the description of the solution smoothness. In the setting of (3.4), $\beta \in (0, 1]$ is a fixed constant, and Ψ is a concave index function. Notice that a function Ψ is called an index function if and only if $\Psi : (0, \infty) \to (0, \infty)$ is continuous and strictly increasing and satisfies $\lim_{t \to 0} \Psi(t) = 0$. As already established in [14, 16], the condition (3.4) implies the following convergence rate:

$$\|w^\delta_{\kappa(\delta)} - w^\dagger\|_W^2 = \mathcal{O}(\Psi(\delta)) \quad \text{as} \quad \delta \to 0. \tag{3.5}$$

In other words, the index function can determine the convergence rate, if the regularization parameter is chosen appropriately. In particular, we have following result:

Proposition 3.1 ([22, theorem 1]). *Let the regularization parameter be chosen a priori as $\kappa = \kappa(\delta) = \frac{\delta^2}{\Psi(\delta)}$. Then, under the variational source condition (3.4), the convergence property (3.5) holds true for the regularized solution $w^\delta_{\kappa(\delta)}$.*

In this paper, we only apply *a priori* rules for choosing the regularization parameter when minimizing T^δ_κ, because the discussion of *a posteriori* rules such as variants of the discrepancy principle or Lepskiĭ principle, which also depend on z^δ, is beyond the scope of this paper. We refer the reader e.g. to [7, 22] for *a posteriori* parameter choice rules under variational source conditions.

3.2. Mathematical properties of (2.3)

We begin by recalling some standard results from the semigroup theory.

Lemma 3.2. *Let assumptions* $(\mathbf{A0})$ *and* $(\mathbf{A1})$ *be satisfied. Then, the mild solution operator* $G : \mathbf{X} \to C([0, T]; \mathbf{X})$ *satisfies*

$$\|G(\mathbf{u}_1, \mathbf{v}_1) - G(\mathbf{u}_2, \mathbf{v}_2)\|_{C([0,T];\mathbf{X})} \lesssim e^{Lt}\|(\mathbf{u}_1, \mathbf{v}_1) - (\mathbf{u}_2, \mathbf{v}_2)\|_{\mathbf{X}} \tag{3.6}$$

for all $(\mathbf{u}_i, \mathbf{v}_i) \in \mathbf{X}$, $i = 1, 2$. *In addition, if* $(\mathbf{u}, \mathbf{v}) \in \mathbf{Y}$, *then* $G(\mathbf{u}, \mathbf{v}) \in C([0, T]; D(\mathcal{A})) \cap C^1([0, T]; \mathbf{X})$ *is the classical solution of the Cauchy problem (2.1).*

Proof. The standard argument in [33, p 184] implies that $G : \mathbf{X} \to C([0, T]; \mathbf{X})$ is well-defined and the estimate (3.6) holds true. Let $(\mathbf{u}, \mathbf{v}) \in \mathbf{Y} \subset D(\mathcal{A})$. Then, in view of **(A1)**, the argument in [33, p 189] ensures that the mild solution $(\mathbf{E}, \mathbf{H}) := G(\mathbf{u}, \mathbf{v})$ is Lipschitz continuous, which implies $\mathbf{f} := \mathbf{F}(\cdot, (\mathbf{E}, \mathbf{H})(\cdot)) \in C^{0,1}([0, T]; \mathbf{X})$. In particular, the reflexivity of $\mathbf{X}$ implies that $\mathbf{f} \in W^{1,\infty}((0, T); \mathbf{X})$. For this reason along with the regularity property $(\mathbf{u}, \mathbf{v}) \in \mathbf{Y} \subset D(\mathcal{A})$, we may apply [9, corollary 7.6, p 440] to deduce that the solution $(\mathbf{E}, \mathbf{H}) \in C([0, T]; D(\mathcal{A})) \cap C^1([0, T]; \mathbf{X})$ is the classical solution. □

Setting $t = T$ into (3.6), we immediately obtain the Lipschitz continuity of the operator $S : \mathbf{X} \to \mathbf{X}$:

$$\|S(\mathbf{u}_1,\mathbf{v}_1)-S(\mathbf{u}_2,\mathbf{v}_2)\|_{\mathbf{X}} \lesssim e^{LT}\|(\mathbf{u}_1,\mathbf{v}_1)-(\mathbf{u}_2,\mathbf{v}_2)\|_{\mathbf{X}} \quad \forall (\mathbf{u}_1,\mathbf{v}_1),(\mathbf{u}_2,\mathbf{v}_2)\in\mathbf{X}. \tag{3.7}$$

In following lemma, we establish another estimate result for the operator $S:\mathbf{X}\to\mathbf{X}$, which is significant for our subsequent analysis.

Proposition 3.3. *Under the assumptions of lemma 3.2, the forward operator S is sequentially weak-to-strong continuous from $\mathbf{Y}$ to $\mathbf{X}$. More importantly, there exists a constant $C_S>0$ such that*

$$\|(\mathbf{u}_1,\mathbf{v}_1)-(\mathbf{u}_2,\mathbf{v}_2)\|_{\mathbf{X}} \leqslant C_S\|S(\mathbf{u}_1,\mathbf{v}_1)-S(\mathbf{u}_2,\mathbf{v}_2)\|_{\mathbf{X}} \quad \forall (\mathbf{u}_1,\mathbf{v}_1),(\mathbf{u}_2,\mathbf{v}_2)\in\mathbf{X}. \tag{3.8}$$

If $\mathbf{F}$, in addition, satisfies the following monotonicity condition:

$$(\mathbf{F}(t,(\mathbf{u}_1,\mathbf{v}_1))-\mathbf{F}(t,(\mathbf{u}_2,\mathbf{v}_2)),(\mathbf{u}_1,\mathbf{v}_1)-(\mathbf{u}_2,\mathbf{v}_2))_{\mathbf{X}}\geqslant 0 \quad \forall t\in[0,T], \tag{3.9}$$

for all $(\mathbf{u}_i,\mathbf{v}_i)\in\mathbf{X}$, $(i=1,2)$, then the estimate (3.8) holds with $C_S=1$.

Proof. Let $\{(\mathbf{u}_n,\mathbf{v}_n)\}_{n=1}^{\infty}\subset\mathbf{Y}$ be a sequence, converging weakly to $(\mathbf{u},\mathbf{v})$ in $\mathbf{Y}$. The compact embedding $\mathbf{Y}\hookrightarrow\mathbf{X}$ (see [29, 34]) implies that $(\mathbf{u}_n,\mathbf{v}_n)\to(\mathbf{u},\mathbf{v})$ strongly in $\mathbf{X}$ as $n\to\infty$. Then, (3.7) implies the strong convergence $S(\mathbf{u}_n,\mathbf{v}_n)\to S(\mathbf{u},\mathbf{v})$ in $\mathbf{X}$. In conclusion, $S:\mathbf{Y}\to\mathbf{X}$ is sequentially weak-to-strong continuous.

Let us now prove the main estimate (3.8). For all $0\leqslant s\leqslant T$, we obtain by the definition of mild solution $(\mathbf{E},\mathbf{H})$ at $t=s$ and $t=T$ that

$$(\mathbf{E},\mathbf{H})(T)=\mathbb{T}_T(\mathbf{E},\mathbf{H})(0)+\int_0^T \mathbb{T}_{T-\tau}\mathbf{F}(\tau,(\mathbf{E},\mathbf{H})(\tau))\mathrm{d}\tau, \tag{3.10}$$

and

$$(\mathbf{E},\mathbf{H})(s)=\mathbb{T}_s(\mathbf{E},\mathbf{H})(0)+\int_0^s \mathbb{T}_{s-\tau}\mathbf{F}(\tau,(\mathbf{E},\mathbf{H})(\tau))\mathrm{d}\tau. \tag{3.11}$$

Applying $\mathbb{T}_{T-s}$ to both sides of (3.11) and taking advantage of the group property of the strongly continuous group $\{\mathbb{T}_t\}_{t\in\mathbb{R}}$, it follows that

$$\mathbb{T}_{T-s}(\mathbf{E},\mathbf{H})(s)=\mathbb{T}_T(\mathbf{E},\mathbf{H})(0)+\int_0^s \mathbb{T}_{T-\tau}\mathbf{F}(\tau,(\mathbf{E},\mathbf{H})(\tau))\mathrm{d}\tau \quad \forall s\in[0,T].$$

In view of this and (3.10), we obtain that

$$(\mathbf{E},\mathbf{H})(T)=\mathbb{T}_{T-s}(\mathbf{E},\mathbf{H})(s)+\int_s^T \mathbb{T}_{T-\tau}\mathbf{F}(\tau,(\mathbf{E},\mathbf{H})(\tau))\mathrm{d}\tau \quad \forall s\in[0,T]. \tag{3.12}$$

Then, applying $\mathbb{T}_{s-T}$ to the above equality yields that

$$(\mathbf{E},\mathbf{H})(s)=\mathbb{T}_{s-T}(\mathbf{E},\mathbf{H})(T)-\int_s^T \mathbb{T}_{s-\tau}\mathbf{F}(\tau,(\mathbf{E},\mathbf{H})(\tau))\mathrm{d}\tau \quad \forall s\in[0,T].$$

Therefore, for two different solution $\mathbf{U}_1 := G(\mathbf{u}_1, \mathbf{v}_1)$ and $\mathbf{U}_2 := G(\mathbf{u}_2, \mathbf{v}_2)$, it holds that

$$(\mathbf{U}_1 - \mathbf{U}_2)(s) = \mathbb{T}_{s-T}(\mathbf{U}_1 - \mathbf{U}_2)(T) - \int_s^T \mathbb{T}_{s-\tau}\left(\mathbf{F}(\tau, \mathbf{U}_1(\tau)) - \mathbf{F}(\tau, \mathbf{U}_2(\tau))\right) \mathrm{d}\tau \quad \forall s \in [0, T].$$

As the group $\{\mathbb{T}_t\}_{t\in\mathbb{R}}$ is unitary and $\mathbf{F}$ is globally Lipschitz-continuous with the Lipschitz constant $L > 0$, we obtain from the above identity that

$$\|(\mathbf{U}_1 - \mathbf{U}_2)(s)\|_{\mathbf{X}} \leqslant \|(\mathbf{U}_1 - \mathbf{U}_2)(T)\|_{\mathbf{X}} + \int_s^T L\|(\mathbf{U}_1 - \mathbf{U}_2)(\tau)\|_{\mathbf{X}} \mathrm{d}\tau \quad \forall s \in [0, T],$$

which implies that

$$\|(\mathbf{U}_1 - \mathbf{U}_2)(T-s)\|_{\mathbf{X}} \leqslant \|(\mathbf{U}_1 - \mathbf{U}_2)(T)\|_{\mathbf{X}} + \int_{T-s}^T L\|(\mathbf{U}_1 - \mathbf{U}_2)(\tau)\|_{\mathbf{X}} \mathrm{d}\tau \quad \forall s \in [0, T].$$

Setting $f(s) := \|(\mathbf{U}_1 - \mathbf{U}_2)(T-s)\|_{\mathbf{X}}$ for all $s \in [0, T]$, we deduce from the above inequality by changing of variables that

$$\begin{aligned} f(s) &\leqslant \|(\mathbf{U}_1 - \mathbf{U}_2)(T)\|_{\mathbf{X}} + \int_0^s L\|(\mathbf{U}_1 - \mathbf{U}_2)(T-w)\|_{\mathbf{X}} \mathrm{d}w \\ &\leqslant \|(\mathbf{U}_1 - \mathbf{U}_2)(T)\|_{\mathbf{X}} + \int_0^s Lf(w) \mathrm{d}w \quad \forall s \in [0, T]. \end{aligned}$$

Thus, the classical Gronwall's inequality implies $f(s) \leqslant \exp(Ls)\|(\mathbf{U}_1 - \mathbf{U}_2)(T)\|$ for all $s \in [0, T]$. Taking $s = T$, we get

$$\|(\mathbf{u}_1, \mathbf{v}_1) - (\mathbf{u}_2, \mathbf{v}_2)\|_{\mathbf{X}} \leqslant \exp(LT)\|(\mathbf{U}_1 - \mathbf{U}_2)(T)\|_{\mathbf{X}} = \exp(LT)\|S(\mathbf{u}_1, \mathbf{v}_1) - S(\mathbf{u}_2, \mathbf{v}_2)\|_{\mathbf{X}}.$$

Now suppose that $\mathbf{F}$ additionally satisfies (3.9). Then, the energy balanced equality implies that

$$\begin{aligned} \|(\mathbf{U}_1 - \mathbf{U}_2)(T)\|_{\mathbf{X}}^2 =& \|(\mathbf{U}_1 - \mathbf{U}_2)(0)\|_{\mathbf{X}}^2 \\ &+ 2\int_0^T (\mathbf{F}(t, \mathbf{U}_1(t)) - \mathbf{F}(t, \mathbf{U}_2(t)), \mathbf{U}_1(t) - \mathbf{U}_2(t))_{\mathbf{X}} \mathrm{d}t, \end{aligned}$$

which due to (3.9) yields that $\|(\mathbf{u}_1, \mathbf{v}_1) - (\mathbf{u}_2, \mathbf{v}_2)\|_{\mathbf{X}} \leqslant \|S(\mathbf{u}_1, \mathbf{v}_1) - S(\mathbf{u}_2, \mathbf{v}_2)\|_{\mathbf{X}}$. □

Based on **(A2)**, proposition 3.3 immediately implies that $(\mathbf{u}^\dagger, \mathbf{v}^\dagger) \in \mathbf{Y}$ is the unique solution to the operator equation: Find $(\mathbf{u}, \mathbf{v}) \in \mathbf{Y}$ such that

$$S(\mathbf{u}, \mathbf{v}) = (\mathbf{e}^\dagger, \mathbf{h}^\dagger) \quad \text{in } \mathbf{X}.$$

Thus, in view of well-known results [8, 31] (see [18, section 4.1] and [32, section 4.1]), proposition 3.3 leads to the following result:

Theorem 3.4. *Assume that the hypothesises (* **A0***)–(* **A2***) hold true.*

(1) For each $\kappa > 0$ and $(\mathbf{E}^\delta, \mathbf{H}^\delta) \in \mathbf{X}$, there exists a minimiser $(\mathbf{u}_\kappa^\delta, \mathbf{v}_\kappa^\delta) \in \mathbf{Y}$ of (2.3).
(2) Let $\{\delta_n\}_{n=1}^\infty \subset \mathbb{R}^+$ be a sequence converging monotonically to zero and $(\mathbf{e}^{\delta_n}, \mathbf{h}^{\delta_n})$ satisfy

$$\|(\mathbf{e}^{\delta_n},\mathbf{h}^{\delta_n})-(\mathbf{e}^{\dagger},\mathbf{h}^{\dagger})\|_{\mathbf{X}}\leqslant\delta_n.$$

Moreover, we assume that the regularization parameter κ_n fulfils

$$\kappa_n\to 0 \text{ and } \frac{\delta_n^2}{\kappa_n}\to 0 \quad \text{as} \quad n\to\infty. \tag{3.13}$$

If $(\mathbf{u}_n,\mathbf{v}_n)$ is a minimiser of (2.3) with $(\mathbf{e}^{\delta},\mathbf{h}^{\delta})$ and κ replaced by $(\mathbf{e}^{\delta_n},\mathbf{h}^{\delta_n})$ and κ_n, respectively, then the sequence $\{(\mathbf{u}_n,\mathbf{v}_n)\}_{n=1}^{\infty}$ converges strongly in $\mathbf{Y}$ to the exact solution $(\mathbf{u}^{\dagger},\mathbf{v}^{\dagger})$ as $n\to\infty$.

3.3. First-order analysis of (2.3)

Our first-order analysis relies on the following assumptions for $\mathbf{F}:[0,T]\times\mathbf{X}\to\mathbf{X}$:

(A3a) For each $t\in[0,T]$, $\mathbf{F}(t,\cdot):\mathbf{X}\to\mathbf{X}$ is Gâteaux-differentiable.
(A3b) The Gâteaux derivative is assumed to satisfy the following property: If

$$t_n\to t \text{ in } [0,T] \quad \text{and} \quad (\mathbf{u}_n,\mathbf{v}_n)\to(\mathbf{u},\mathbf{v})\text{strongly in } \mathbf{X},$$

then for every $(\mathbf{w},\mathbf{y})\in\mathbf{X}$ it holds that

$$\lim_{n\to\infty}\|\mathbf{F}'(t_n,\mathbf{u}_n,\mathbf{v}_n)(\mathbf{w},\mathbf{y})-\mathbf{F}'(t,\mathbf{u},\mathbf{v})(\mathbf{w},\mathbf{y})\|_{\mathbf{X}}=0.$$

(A3c) The function $(t,\mathbf{u},\mathbf{v})\mapsto\mathbf{F}'(t,\mathbf{u},\mathbf{v})$ maps every bounded set in $[0,T]\times\mathbf{X}$ into a bounded set in $\mathcal{L}(\mathbf{X})$.

Lemma 3.5. *Let (* **A0***)–(* **A1***) and (* **A3***) be satisfied. Then, the operator $S:\mathbf{X}\to\mathbf{X}$ is weakly directionally differentiable. The weak directional derivative of S at $(\mathbf{u},\mathbf{v})\in\mathbf{X}$ in the direction $(\delta\mathbf{u},\delta\mathbf{v})\in\mathbf{X}$ is given $S'(\mathbf{u},\mathbf{v})(\delta\mathbf{u},\delta\mathbf{v})=(\delta\mathbf{E},\delta\mathbf{H})(T)$, where $(\delta\mathbf{E},\delta\mathbf{H})\in C([0,T];\mathbf{X})$ satisfies the following integral equation*

$$(\delta\mathbf{E},\delta\mathbf{H})(t)=\mathbb{T}_t(\delta\mathbf{u},\delta\mathbf{v})+\int_0^t\mathbb{T}_{t-s}\mathbf{F}'(s,(\mathbf{E},\mathbf{H})(s))(\delta\mathbf{E},\delta\mathbf{H})(s)\mathrm{d}s\quad\forall\, t\in[0,T]. \tag{3.14}$$

Proof. Let $(\mathbf{u},\mathbf{v})$, $(\delta\mathbf{u},\delta\mathbf{v})\in\mathbf{X}$ and $(\mathbf{E},\mathbf{H})=G(\mathbf{u},\mathbf{v})$. Further, for every $\tau\in\mathbb{R}^+$, we write $(\mathbf{E}_\tau,\mathbf{H}_\tau)=G(\mathbf{u}+\tau\delta\mathbf{u},\mathbf{v}+\tau\delta\mathbf{v})$. Thus, according to (3.6), we have

$$\left(\frac{\mathbf{E}_\tau-\mathbf{E}}{\tau},\frac{\mathbf{H}_\tau-\mathbf{H}}{\tau}\right)(t)=\mathbb{T}_t(\delta\mathbf{u},\delta\mathbf{v})+\int_0^t\mathbb{T}_{t-s}I_\tau(s)\mathrm{d}s, \tag{3.15}$$

where

$$I_\tau(s):=\frac{\mathbf{F}(s,(\mathbf{E}_\tau,\mathbf{H}_\tau)(s))-\mathbf{F}(s,(\mathbf{E},\mathbf{H})(s))}{\tau}. \tag{3.16}$$

Lemma 3.2 implies that $\{(\frac{\mathbf{E}_\tau-\mathbf{E}}{\tau},\frac{\mathbf{H}_\tau-\mathbf{H}}{\tau})\}_{\tau>0}$ is bounded in $C([0,T];\mathbf{X})$, and hence there exists a subsequence, which we still denote by $\{(\frac{\mathbf{E}_\tau-\mathbf{E}}{\tau},\frac{\mathbf{H}_\tau-\mathbf{H}}{\tau})\}_{\tau>0}$, such that

$$\left(\frac{\mathbf{E}_\tau-\mathbf{E}}{\tau},\frac{\mathbf{H}_\tau-\mathbf{H}}{\tau}\right)\rightharpoonup(\delta\mathbf{E},\delta\mathbf{H})\text{ weakly star in } L^\infty((0,T),\mathbf{X})\text{ as }\tau\to 0^+, \tag{3.17}$$

for some $(\delta\mathbf{E},\delta\mathbf{H})\in L^\infty((0,T),\mathbf{X})$.

For the sake of brevity, we write $\mathbf{U}_\tau = (\mathbf{E}_\tau, \mathbf{H}_\tau)$ and $\mathbf{U} = (\mathbf{E}, \mathbf{H})$. Let $\mathbf{w} \in L^1((0,T);\mathbf{X})$ be arbitrarily fixed. The mean-value theorem in the integral form implies that for almost all $s \in (0,T)$,

$$\begin{aligned}&(I_\tau(s), \mathbf{w}(s))_\mathbf{X}\\ =&(\mathbf{F}'(s,\mathbf{U}(s))\frac{\mathbf{U}_\tau(s)-\mathbf{U}(s)}{\tau}, \mathbf{w}(s))_\mathbf{X} + (\mathbf{G}_\tau(s)\frac{\mathbf{U}_\tau(s)-\mathbf{U}(s)}{\tau}, \mathbf{w}(s))_\mathbf{X},\end{aligned} \tag{3.18}$$

where

$$\mathbf{G}_\tau(s)\boldsymbol{x} := \int_0^1 \mathbf{F}'(s,(\mathbf{U}+\theta(\mathbf{U}_\tau-\mathbf{U}))(s))\boldsymbol{x} - \mathbf{F}'(s,\mathbf{U}(s))\boldsymbol{x}\mathrm{d}\theta \quad \forall \boldsymbol{x} \in \mathbf{X}.$$

As $\mathbf{U} \in C([0,T];\mathbf{X})$ and $\{\mathbf{U}_\tau\}_{\tau>0} \subset C([0,T];\mathbf{X})$ is bounded, **(A3c)** implies the existence of a positive constant $M > 0$, independent of t and τ, such that

$$\|\mathbf{G}_\tau^*(t)\|_{\mathcal{L}(\mathbf{X})} = \|\mathbf{G}_\tau(t)\|_{\mathcal{L}(\mathbf{X})} \leqslant M \quad \forall t \in [0,T], \quad \forall \tau \in \mathbb{R}^+. \tag{3.19}$$

Also, by **(A3b)**, it holds that

$$\mathbf{G}_\tau(\cdot)\boldsymbol{x} \in \mathcal{C}([0,T];\mathbf{X}) \quad \forall \boldsymbol{x} \in \mathbf{X}, \quad \forall \tau \in \mathbb{R}^+. \tag{3.20}$$

Therefore, (3.19) and (3.20) along with Petti's theorem and the separability of $\mathbf{X}$ imply that $\mathbf{G}_\tau(\cdot)^*\mathbf{w}(\cdot)$ belongs to $L^1((0,T);\mathbf{X})$. Moreover, since for almost every $s \in (0,T)$ $\mathbf{G}_\tau(s)^*\mathbf{w}(s) \to 0$ as $\tau \to 0^+$ and by (3.19), we may apply Lebesgue's dominated convergence theorem to obtain that $\mathbf{G}_\tau^*\mathbf{w} \to 0$ strongly in $L^1((0,T);\mathbf{X})$ as $\tau \to 0^+$. Then, the weak star convergence (3.17) implies

$$\int_0^T (\mathbf{G}_\tau(s)\frac{\mathbf{U}_\tau(s)-\mathbf{U}(s)}{\tau}, \mathbf{w}(s))_\mathbf{X}\mathrm{d}s = \int_0^T (\frac{\mathbf{U}_\tau(s)-\mathbf{U}(s)}{\tau}, \mathbf{G}_\tau^*(s)\mathbf{w}(s))_\mathbf{X}\mathrm{d}s \to 0, \tag{3.21}$$

as $\tau \to 0^+$. In addition, (3.17) along with **(A3c)** yields

$$\lim_{\tau\to 0}\int_0^T (\mathbf{F}'(s,\mathbf{U}(s))\frac{\mathbf{U}_\tau(s)-\mathbf{U}(s)}{\tau}, \mathbf{w}(s))_\mathbf{X}\mathrm{d}s = \int_0^T (\mathbf{F}'(s,\mathbf{U}(s))(\delta\mathbf{E}(s),\delta\mathbf{H}(s)), \mathbf{w}(s))_\mathbf{X}\,\mathrm{d}s. \tag{3.22}$$

Concluding from (3.18), (3.21), (3.22) and since $\mathbf{w} \in L^1((0,T);\mathbf{X})$ was chosen arbitrarily fixed, we obtain that

$$I_\tau \rightharpoonup \mathbf{F}'(\cdot,\mathbf{U}(\cdot))(\delta\mathbf{E},\delta\mathbf{H}) \quad \text{weakly star in } L^\infty((0,T);\mathbf{X}) \text{ as } \tau \to 0^+. \tag{3.23}$$

On the other hand, it holds that

$$\int_0^T\int_0^t (\mathbb{T}_{t-s}I_\tau(s), \mathbf{w}(t))_\mathbf{X}\,\mathrm{d}s\mathrm{d}t = \int_0^T \left(I_\tau(s), \int_s^T \mathbb{T}_{t-s}^*\mathbf{w}(\mathbf{t})\mathrm{d}t\right)_\mathbf{X}\mathrm{d}s \quad \forall\, \mathbf{w} \in L^1((0,T);\mathbf{X}).$$

Since the mapping $s \mapsto \int_s^T \mathbb{T}_{t-s}^*\mathbf{w}(t)\mathrm{d}t$ also belongs to $L^1(0,T;\mathbf{X})$, we obtain from (3.23), (3.17) and (3.15) that $(\delta\mathbf{E},\delta\mathbf{H})$ satisfies the integral equation (3.14). □

Corollary 3.6. *Under the assumptions of lemma 3.5, the operator* $S:\mathbf{X}\to\mathbf{X}$ *is weakly-Gâteaux-differentiable.*

Proof. Let $(\mathbf{u},\mathbf{v})\in\mathbf{X}$ and $(\mathbf{E},\mathbf{H})=G(\mathbf{u},\mathbf{v})\in C([0,T];\mathbf{X})$. In view of (3.14), the operator $S'(\mathbf{u},\mathbf{v}):\mathbf{X}\to\mathbf{X}$ is linear. Let us show the boundedness result. In fact, the energy balance equality implies that $(\delta\mathbf{E},\delta\mathbf{H})=G(\mathbf{u},\mathbf{v})(\delta\mathbf{u},\delta\mathbf{v})$ satisfies

$$\begin{aligned}&\|(\delta\mathbf{E}(t),\delta\mathbf{H}(t))\|_{\mathbf{X}}^2=\|(\delta\mathbf{u},\delta\mathbf{v})\|_{X}^2\\&+2\int_0^t(\mathbf{F}'(s,(\mathbf{E},\mathbf{H})(s))(\delta\mathbf{E},\delta\mathbf{H})(s),(\delta\mathbf{E},\delta\mathbf{H})(s))_{\mathbf{X}}\,\mathrm{d}s\quad\forall\,t\in[0,T].\end{aligned}$$

Then, according to **(A3c)**, we can find a constant $C>0$ independent of $(\delta\mathbf{E},\delta\mathbf{H})$ such that

$$\|(\delta\mathbf{E},\delta\mathbf{H})(t)\|_{\mathbf{X}}^2\leqslant\|(\delta\mathbf{u},\delta\mathbf{v})\|_{X}^2+C\int_0^t\|(\delta\mathbf{E},\delta\mathbf{H})(s)\|_{\mathbf{X}}^2\mathrm{d}s\quad\forall\,t\in[0,T].$$

From this inequality, the Gronwall lemma implies that $S'(\mathbf{u},\mathbf{v}):\mathbf{X}\to\mathbf{X}$ is bounded. □

By classical arguments, corollary 3.6 implies the following Gâteaux-differentiability result:

Corollary 3.7. *Under the assumptions of lemma 3.5, the functional* $\mathcal{J}_\delta^\kappa:\mathbf{Y}\to\mathbb{R}$ *is Gâteaux-differentiable with the Gâteaux derivative*

$$\mathcal{J}_\delta^{\kappa\prime}(\mathbf{u},\mathbf{v})(\delta\mathbf{u},\delta\mathbf{v})=(S(\mathbf{u},\mathbf{v})-(\mathbf{E}^\delta,\mathbf{H}^\delta),S'(\mathbf{u},\mathbf{v})(\delta\mathbf{u},\delta\mathbf{v}))_{\mathbf{X}}+\kappa((\mathbf{u},\mathbf{v}),(\delta\mathbf{u},\delta\mathbf{v}))_{\mathbf{Y}},\tag{3.24}$$

for all $(\mathbf{u},\mathbf{v}),(\delta\mathbf{u},\delta\mathbf{v})\in\mathbf{Y}$.

Let us now establish an explicit formula for the adjoint operator associated with the Gâteaux-derivative $S'(\mathbf{u},\mathbf{v}):\mathbf{X}\to\mathbf{X}$.

Lemma 3.8. *Let the assumptions of lemma 3.5 be satisfied. Let* $(\mathbf{u},\mathbf{v})\in\mathbf{X}$ *and* $(\mathbf{E},\mathbf{H})\in C([0,T];\mathbf{X})$ *denote the mild solution associated with the initial value* $(\mathbf{u},\mathbf{v})$. *Then, for every* $(\mathbf{w},\mathbf{z})\in\mathbf{X}$, *the adjoint operator* $S'(\mathbf{u},\mathbf{v})^*:\mathbf{X}\to\mathbf{X}$ *satisfies*

$$S'(\mathbf{u},\mathbf{v})^*(\mathbf{w},\mathbf{z}):=(\mathbf{K},\mathbf{Q})(0),$$

where $(\mathbf{K},\mathbf{Q})\in C([0,T];\mathbf{X})$ *is the unique solution of the following integral equation:*

$$(\mathbf{K},\mathbf{Q})(t)=\mathbb{T}_{t-T}(\mathbf{w},\mathbf{z})+\int_t^T\mathbb{T}_{t-s}\mathbf{F}'(s,(\mathbf{E},\mathbf{H})(s))^*(\mathbf{K},\mathbf{Q})(s)\mathrm{d}s\tag{3.25}$$

for all $t\in[0,T]$.

Proof. Let $B(t):=\mathbf{F}'(t,\mathbf{E}(t),\mathbf{H}(t))$ for all $t\in[0,T]$. From **(A3b)** together with $(\mathbf{E},\mathbf{H})\in C([0,T];\mathbf{X})$, it follows that

$$B(\cdot)\boldsymbol{x}\in C([0,T];\mathbf{X}),\quad B(\cdot)\mathbf{w}(\cdot)\in L^1((0,T);\mathbf{X})\quad\forall\,\boldsymbol{x}\in\mathbf{X},\ \mathbf{w}\in L^1((0,T);\mathbf{X}).\tag{3.26}$$

Let $(\hat{\mathbf{w}},\hat{\mathbf{z}})\in D(\mathcal{A})$ and

$$(\delta\mathbf{E},\delta\mathbf{H})(t):=\mathbb{T}_t(\hat{\mathbf{w}},\hat{\mathbf{z}})+\int_0^t\mathbb{T}_{t-s}B(s)(\delta\mathbf{E},\delta\mathbf{H})(s)\mathrm{d}s\quad\forall\,t\in[0,T].\tag{3.27}$$

According to lemma 3.5,

$$S'(\mathbf{u},\mathbf{v})(\hat{\mathbf{w}},\hat{\mathbf{z}}) = (\delta\mathbf{E},\delta\mathbf{H})(T). \tag{3.28}$$

Further, making use of the resolvent operator of $\mathcal{A}$, we introduce the following approximation:

$$B_n(t) := n(nI_d - \mathcal{A})^{-1}B(t) \quad \forall\, n\in\mathbb{N}, \quad \forall\, t\in[0,T]. \tag{3.29}$$

It is standard to show that, for every $n\in\mathbb{N}$, $n(nI_d-\mathcal{A})^{-1}:\mathbf{X}\to D(\mathcal{A})$ is linear and bounded. Thus, along with (3.26), it holds for every $n\in\mathbb{N}$ that

$$B_n(\cdot)\boldsymbol{x}\in C([0,T];D(\mathcal{A})), \quad B_n(\cdot)\mathbf{w}(\cdot)\in L^1((0,T);D(\mathcal{A})) \quad \forall\,\boldsymbol{x}\in\mathbf{X},\ \mathbf{w}\in L^1((0,T);\mathbf{X}).$$

For this reason, the integral equation

$$(\delta\mathbf{E}_n,\delta\mathbf{H}_n)(t) = \mathbb{T}_t(\hat{\mathbf{w}},\hat{\mathbf{z}}) + \int_0^t \mathbb{T}_{t-s}B_n(s)(\delta\mathbf{E}_n,\delta\mathbf{H}_n)(s)\mathrm{d}s \quad \forall\, t\in[0,T] \tag{3.30}$$

admits a unique solution $(\delta\mathbf{E}_n,\delta\mathbf{H}_n)\in\mathcal{C}^1([0,T];\mathbf{X})\cap\mathcal{C}([0,T];D(\mathcal{A}))$ satisfying

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(\delta\mathbf{E}_n,\delta\mathbf{H}_n)(t) = (\mathcal{A}+B_n(t))(\delta\mathbf{E}_n,\delta\mathbf{H}_n)(t) \quad \forall t\in[0,T] \\ (\delta\mathbf{E}_n,\delta\mathbf{H}_n)(0) = (\hat{\mathbf{w}},\hat{\mathbf{z}}). \end{cases} \tag{3.31}$$

Similarly, for every $(\mathbf{w},\mathbf{z})\in D(\mathcal{A})$, the following integral equation

$$(\mathbf{K}_n,\mathbf{Q}_n)(t) = \mathbb{T}_{t-T}(\mathbf{w},\mathbf{z}) + \int_t^T \mathbb{T}_{t-s}B_n(s)^*(\mathbf{K}_n,\mathbf{Q}_n)(s)\mathrm{d}s \quad \forall\, t\in[0,T]$$

admits a unique solution $(\mathbf{K}_n,\mathbf{Q}_n)\in\mathcal{C}^1([0,T];\mathbf{X})\cap\mathcal{C}([0,T];D(\mathcal{A}))$ satisfying

$$\begin{cases} -\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{K}_n,\mathbf{Q}_n)(t) = (-\mathcal{A}+B_n(t)^*)(\mathbf{K}_n,\mathbf{Q}_n)(t) \quad \forall t\in[0,T] \\ (\mathbf{K}_n,\mathbf{Q}_n)(T) = (\mathbf{w},\mathbf{z}). \end{cases} \tag{3.32}$$

Combining (3.31) and (3.32), we obtain that

$$\begin{aligned} &((\delta\mathbf{E}_n,\delta\mathbf{H}_n)(T),(\mathbf{K}_n,\mathbf{Q}_n)(T))_{\mathbf{X}} - ((\delta\mathbf{E}_n,\delta\mathbf{H}_n)(0),(\mathbf{K}_n,\mathbf{Q}_n)(0))_{\mathbf{X}} \\ &= \int_0^T \big(\frac{\mathrm{d}}{\mathrm{d}t}(\delta\mathbf{E}_n,\delta\mathbf{H}_n)(t),(\mathbf{K}_n,\mathbf{Q}_n)(t)\big)_{\mathbf{X}} + \big((\delta\mathbf{E}_n,\delta\mathbf{H}_n)(t),\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{K}_n,\mathbf{Q}_n)(t)\big)_{\mathbf{X}}\mathrm{d}t \\ &= \int_0^T \big((\delta\mathbf{E}_n,\delta\mathbf{H}_n)(t),(\mathcal{A}+B_n(t))^*(\mathbf{K}_n,\mathbf{Q}_n)(t)+\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{K}_n,\mathbf{Q}_n)(t)\big)_{\mathbf{X}}\mathrm{d}t = 0. \end{aligned} \tag{3.33}$$

A direct computation based on (3.27) and (3.30) yields

$$\begin{aligned} &\|(\delta\mathbf{E}_n,\delta\mathbf{H}_n)(t)-(\delta\mathbf{E},\delta\mathbf{H})(t)\|_{\mathbf{X}} \\ &\leqslant \Big\|\int_0^t \mathbb{T}_{t-s}B_n(s)((\delta\mathbf{E}_n,\delta\mathbf{H}_n)(s)-(\delta\mathbf{E},\delta\mathbf{H})(s))\mathrm{d}s\Big\|_{\mathbf{X}} \\ &+ \Big\|\int_0^t \mathbb{T}_{t-s}(B_n(s)-B(s))(\delta\mathbf{E},\delta\mathbf{H})(s)\mathrm{d}s\Big\|_{\mathbf{X}} \quad \forall\, t\in[0,T]. \end{aligned} \tag{3.34}$$

Since for every $\boldsymbol{x} \in \mathbf{X}$ the mapping $t \mapsto B(t)\boldsymbol{x}$ is continuous from $[0, T]$ to $\mathbf{X}$ and the set $\{(\delta\mathbf{E}, \delta\mathbf{H})(t) \mid t \in [0, T]\}$ is compact in $\mathbf{X}$, it follows that (see [9, lemma 5.2, chapter 1]) the mapping $t \mapsto B(t)(\delta\mathbf{E}, \delta\mathbf{H})(t)$ is continuous from $[0, T]$ to $\mathbf{X}$, and consequently the set $\{B(t)(\delta\mathbf{E}, \delta\mathbf{H})(t) \mid t \in [0, T]\}$ is a compact set in $\mathbf{X}$. Then, as $\lim_{n\to\infty} n(nI_d - \mathcal{A})^{-1}\boldsymbol{x} = \boldsymbol{x}$ holds for all $\boldsymbol{x} \in \mathbf{X}$ (see [9, lemma 3.4, chapter 2]), we obtain again by employing [9, lemma 5.2c, chapter 1] that

$$\forall \epsilon > 0\, \exists N \in \mathbb{N}\ \forall n \geqslant N : \max_{s\in[0,T]} \| \underbrace{(B_n(s) - B(s))}_{=(n(nI_d-\mathcal{A})^{-1}-I_d)B(s)} (\delta\mathbf{E}, \delta\mathbf{H})(s)\|_{\mathbf{X}} \leqslant \varepsilon. \tag{3.35}$$

On the other hand, by definition (3.29) and **(A3c)**, there exists a constant $C_B > 0$, independent of n, such that

$$\sup_{t\in[0,T]} \|B_n(t)\|_{\mathcal{L}(\mathbf{X})} \leqslant C_B \quad \forall\, n \in \mathbb{N}. \tag{3.36}$$

Therefore, in view of (3.34)–(3.36), it holds that for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$\begin{aligned}&\|(\delta\mathbf{E}_n, \delta\mathbf{H}_n)(t) - (\delta\mathbf{E}, \delta\mathbf{H})(t)\|_{\mathbf{X}} \\ \leqslant& C_B \int_0^t \|(\delta\mathbf{E}_n, \delta\mathbf{H}_n)(s) - (\delta\mathbf{E}, \delta\mathbf{H})(s)\|_{\mathbf{X}} + t\epsilon \quad \forall\, t \in [0, T],\ \forall\, n \geqslant N.\end{aligned}$$

Hence, the Gronwall lemma implies that

$$\lim_{n\to\infty} (\delta\mathbf{E}_n, \delta\mathbf{H}_n)(t) = (\delta\mathbf{E}, \delta\mathbf{H})(t) \quad \forall t \in [0, T]. \tag{3.37}$$

Similarly, one can show that

$$\lim_{n\to\infty} (\mathbf{K}_n, \mathbf{Q}_n)(t) = (\mathbf{K}, \mathbf{Q})(t) \quad \forall t \in [0, T], \tag{3.38}$$

where $(\mathbf{K}, \mathbf{Q})$ is the solution of the integral solution (3.25). Combining (3.28), (3.33), (3.37) and (3.38), we obtain that

$$(S'(\mathbf{u}, \mathbf{v})(\hat{\mathbf{w}}, \hat{\mathbf{z}}), (\mathbf{w}, \mathbf{z}))_{\mathbf{X}} = ((\hat{\mathbf{w}}, \hat{\mathbf{z}}), (\mathbf{K}, \mathbf{Q})(0))_{\mathbf{X}}$$

for all $(\mathbf{w}, \mathbf{z}), (\hat{\mathbf{w}}, \hat{\mathbf{z}}) \in D(\mathcal{A})$. Then, by the density of $D(\mathcal{A})$ in $\mathbf{X}$, the assertion is valid. □

Theorem 3.9. *Assume that (* **A0***)–(* **A1***) and* (**A3**) *hold true. Furthermore, let* $(\mathbf{u}^\delta_\kappa, \mathbf{v}^\delta_\kappa) \in \mathbf{Y}$ *be a minimiser of (2.3) and* $(\mathbf{E}^\delta_\kappa, \mathbf{H}^\delta_\kappa) \in C([0, T]; D(\mathcal{A})) \cap C^1([0, T]; \mathbf{X})$ *the corresponding solution of (2.1) associated with* $(\mathbf{u}^\delta_\kappa, \mathbf{v}^\delta_\kappa) \in \mathbf{Y}$*. Then, there exists a unique adjoint state* $(\mathbf{K}^\delta_\kappa, \mathbf{Q}^\delta_\kappa) \in C([0, T]; \mathbf{X})$ *satisfying*

$$\begin{aligned}(\mathbf{K}^\delta_\kappa, \mathbf{Q}^\delta_\kappa)(t) =& \mathbb{T}_{t-T}(\mathbf{E}^\delta_\kappa(T) - \mathbf{e}^\delta, \mathbf{H}^\delta_\kappa(T) - \mathbf{h}^\delta), \\ &+ \int_t^T \mathbb{T}_{t-s}\mathbf{F}'(s, (\mathbf{E}^\delta_\kappa, \mathbf{H}^\delta_\kappa)(s))^*(\mathbf{K}^\delta_\kappa, \mathbf{Q}^\delta_\kappa)(s)\mathrm{d}s \quad \forall t \in [0, T]\end{aligned} \tag{3.39}$$

$$(\kappa(\mathbf{u}^\delta_\kappa, \mathbf{v}^\delta_\kappa), (\delta\mathbf{u}, \delta\mathbf{v}))_{\mathbf{Y}} = -((\mathbf{K}^\delta_\kappa, \mathbf{Q}^\delta_\kappa)(0), (\delta\mathbf{u}, \delta\mathbf{v}))_{\mathbf{X}} \quad \forall(\delta\mathbf{u}, \delta\mathbf{v}) \in \mathbf{Y}. \tag{3.40}$$

Proof. The necessary optimality condition for (2.3) reads as

$$\mathcal{J}_\delta^{\kappa\prime}(\mathbf{u}_\kappa^\delta, \mathbf{v}_\kappa^\delta)(\delta\mathbf{u}, \delta\mathbf{v}) = 0, \quad \forall\,(\delta\mathbf{u}, \delta\mathbf{v}) \in \mathbf{Y},$$

which is according to (3.24) equivalent to

$$(S(\mathbf{u}_\kappa^\delta, \mathbf{v}_\kappa^\delta) - (\mathbf{E}^\delta, \mathbf{H}^\delta), S'(\mathbf{u}_\kappa^\delta, \mathbf{v}_\kappa^\delta)(\delta\mathbf{u}, \delta\mathbf{v}))_{\mathbf{X}} + (\kappa(\mathbf{u}_\kappa^\delta, \mathbf{v}_\kappa^\delta), (\delta\mathbf{u}, \delta\mathbf{v}))_{\mathbf{Y}} = 0, \quad \forall\,(\delta\mathbf{u}, \delta\mathbf{v}) \in \mathbf{Y}.$$

Thus, by lemma 3.8, we obtain the desired result. □

Corollary 3.10. *Let* (**A0**)–(**A3**) *be satisfied. Let* $\{\delta_n\}_{n=1}^\infty, \{\kappa_n\}_{n=1}^\infty \subset \mathbb{R}$, $\{(\mathbf{e}^{\delta_n}, \mathbf{h}^{\delta_n})\}_{n=1}^\infty \subset \mathbf{X}$, *and* $\{\mathbf{u}_n, \mathbf{v}_n\}_{n=1}^\infty \subset \mathbf{Y}$ *be sequences as defined in theorem 3.4. Moreover, for every* $n \in \mathbb{N}$, *let* $(\mathbf{E}_{\kappa_n}^{\delta_n}, \mathbf{H}_{\kappa_n}^{\delta_n}) \in C([0,T]; D(\mathcal{A})) \cap C^1([0,T]; \mathbf{X})$ *denote the corresponding solution of (2.1) associated with* $(\mathbf{u}_n, \mathbf{v}_n) \in \mathbf{Y}$, *and* $(\mathbf{K}_n, \mathbf{Q}_n) \in C([0,T]; \mathbf{X})$ *denote the adjoint state satisfying (3.39) and (3.40) with* $(\mathbf{E}_\kappa^\delta, \mathbf{H}_\kappa^\delta)$, $(\mathbf{e}^\delta, \mathbf{h}^\delta)$ *and* $(\mathbf{u}_\kappa^\delta, \mathbf{v}_\kappa^\delta)$ *replaced by* $(\mathbf{E}_{\kappa_n}^{\delta_n}, \mathbf{H}_{\kappa_n}^{\delta_n})$, $(\mathbf{e}^{\delta_n}, \mathbf{h}^{\delta_n})$ *and* $(\mathbf{u}_n, \mathbf{v}_n)$, *respectively. Then, the adjoint state* $(\mathbf{K}_n, \mathbf{Q}_n)$ *converges strongly in* $C([0,T]; \mathbf{X})$ *to zero as* $n \to \infty$.

Proof. From theorem 3.4, it follows that $(\mathbf{u}_n, \mathbf{v}_n)$ converges strongly to $(\mathbf{u}^\dagger, \mathbf{v}^\dagger)$. The Lipschtiz continuity of $S : \mathbf{X} \to \mathbf{X}$ implies that $(\mathbf{E}_{\kappa_n}^{\delta_n}, \mathbf{H}_{\kappa_n}^{\delta_n})(T)$ converges strongly to $(\mathbf{e}^\dagger, \mathbf{h}^\dagger)$. As a consequence, we obtain that

$$\lim_{n\to\infty} \|(\mathbf{E}_{\kappa_n}^{\delta_n}(T) - \mathbf{e}^{\delta_n}, \mathbf{H}_{\kappa_n}^{\delta_n}(T) - \mathbf{h}^{\delta_n})\|_{\mathbf{X}} = 0. \tag{3.41}$$

On the other hand, since $\{(\mathbf{E}_n, \mathbf{H}_n)\}_{n=1}^\infty \subset C([0,T]; X)$ is bounded, we may apply the assumption (**A3c**) to the equation (3.39) and obtain a constant $M > 0$ such that

$$\begin{aligned} &\|(\mathbf{K}_n, \mathbf{Q}_n)(t)\|_{\mathbf{X}} \\ &\leqslant \|(\mathbf{E}_{\kappa_n}^{\delta_n}(T) - \mathbf{e}^{\delta_n}, \mathbf{H}_{\kappa_n}^{\delta_n}(T) - \mathbf{h}^{\delta_n})\|_{\mathbf{X}} + M\int_t^T \|(\mathbf{K}_n(s), \mathbf{Q}_n(s))\|_{\mathbf{X}} \mathrm{d}s \quad \forall t \in [0,T]. \end{aligned}$$

The Gronwall lemma implies therefore

$$\|(\mathbf{K}_n, \mathbf{Q}_n)(t)\|_{\mathbf{X}} \leqslant e^{M(T-t)} \|(\mathbf{E}_{\kappa_n}^{\delta_n}(T) - \mathbf{e}^{\delta_n}, \mathbf{H}_{\kappa_n}^{\delta_n}(T) - \mathbf{h}^{\delta_n})\|_{\mathbf{X}} \quad \forall t \in [0,T]. \tag{3.42}$$

Now, the assertion follows from (3.41) and (3.42). □

4. Convergence rate analysis under VSC

As pointed in section 3.1, the convergence speed for the Tikhonov regularization method (2.3) can be arbitrarily slow (see [8]). The goal of this section is therefore to analyze the convergence rate for regularized solutions to (2.3) under VSC (see proposition 3.1). To be more precise, our goal is to to verify

$$\begin{aligned} \frac{\beta}{2}\|(\mathbf{u}^\dagger - \mathbf{u}, \mathbf{v}^\dagger - \mathbf{v})\|_{\mathbf{Y}}^2 \leqslant & \frac{1}{2}\|(\mathbf{u}, \mathbf{v})\|_{\mathbf{Y}}^2 - \frac{1}{2}\|(\mathbf{u}^\dagger, \mathbf{v}^\dagger)\|_{\mathbf{Y}}^2 \\ & + \Psi(\|S(\mathbf{u}^\dagger, \mathbf{v}^\dagger) - S(\mathbf{u}, \mathbf{v})\|_{\mathbf{X}}) \quad \forall\,(\mathbf{u}, \mathbf{v}) \in \mathbf{Y}, \end{aligned} \tag{4.1}$$

for some constant $0 < \beta \leqslant 1$ and index function Ψ. To this end, we shall first establish several auxiliary results in section 4.1 and recall well-known results on interpolation spaces. Then, the verification of (4.1) will be investigated in section 4.2. Throughout this section, we make the following additional material assumption:

(**A4**) There exist Lipschitz domains $\Omega_j \subset \mathbb{R}^3,\ j = 1, \ldots, N$ such that

$$\Omega_i \cap \Omega_j = \emptyset \quad \forall\, i \neq j \quad \text{and} \quad \overline{\Omega} = \cup_{j=1}^{N} \overline{\Omega}_j,$$

and

$$\varepsilon_{|\Omega_j}, \mu_{|\Omega_j} \in C^2(\overline{\Omega}_j)^{3\times 3} \quad \forall j = 1, 2, \ldots, N. \tag{4.2}$$

In order to model a heterogeneous medium, the assumption of piecewise smooth material functions (4.2) is reasonable and often used in the mathematical study of Maxwell's equations (see, e.g. [28, p 83]). Such piecewise smooth assumption was also considered in the study of pulsed electric fields on the physical media involving a heterogeneous permittivity and a heterogeneous conductivity [1].

Lemma 4.1. *Let* (**A0**) *and* (**A4**) *be satisfied. Then the space* **Y** *is dense in* **X**.

Proof. Let us consider the linear spaces

$$\mathcal{D} := \{\mathbf{u} \in C_0^\infty(\Omega)^3; \mathbf{u}\,|_{\Omega_i} \in C_0^\infty(\Omega_i)^3 \text{ for all } \ i = 1, 2, \cdots, N\}. \tag{4.3}$$

Thanks to (4.2), it holds that

$$\mathcal{D} \subset \mathbf{H_0}(\mathbf{curl}) \cap \varepsilon^{-1}\mathbf{H}(\mathrm{div}), \quad \mathcal{D} \subset \mathbf{H}(\mathbf{curl}) \cap \mu^{-1}\mathbf{H_0}(\mathrm{div}),$$

from which it follows that $\mathcal{D} \times \mathcal{D} \subset \mathbf{Y}$. Moreover, by the construction, $\mathcal{D} \times \mathcal{D}$ is dense in $\mathbf{X}$. This completes the proof. □

4.1. An auxiliary result

In the following, we investigate the connection between the inner products of **X** and **Y**, which will be characterized by an unbounded self-adjoint operator. To take advantage of the spectral theory for operators in complex Hilbert spaces and the complex interpolation theory, we need to consider the complexification of **X** and **Y**. More precisely, let $\mathbf{X}_\mathbb{C}$ be a complex linear space consists of all complex-valued functions $(\mathbf{u}, \mathbf{v})$ with $(\mathrm{Re}\,\mathbf{u}, \mathrm{Re}\,\mathbf{v}), (\mathrm{Im}\,\mathbf{u}, \mathrm{Im}\,\mathbf{v}) \in \mathbf{X}$, equipped with inner product

$$\begin{aligned}((\mathbf{u}_1, \mathbf{v}_1), (\mathbf{u}_2, \mathbf{v}_2))_{\mathbf{X}_\mathbb{C}} =& (\mathrm{Re}(\mathbf{u}_1, \mathbf{v}_1), \mathrm{Re}(\mathbf{u}_2, \mathbf{v}_2))_\mathbf{X} + (\mathrm{Im}(\mathbf{u}_1, \mathbf{v}_1), \mathrm{Im}(\mathbf{u}_2, \mathbf{v}_2))_\mathbf{X} \\ &- \mathrm{i}(\mathrm{Re}(\mathbf{u}_1, \mathbf{v}_1), \mathrm{Im}(\mathbf{u}_2, \mathbf{v}_2))_\mathbf{X} + \mathrm{i}(\mathrm{Im}(\mathbf{u}_1, \mathbf{v}_1), \mathrm{Re}(\mathbf{u}_2, \mathbf{v}_2))_\mathbf{X}.\end{aligned}$$

It is obvious that $((\mathbf{u}_1, \mathbf{v}_1), (\mathbf{u}_2, \mathbf{v}_2))_{\mathbf{X}_\mathbb{C}} = ((\mathbf{u}_1, \mathbf{v}_1), (\mathbf{u}_2, \mathbf{v}_2))_\mathbf{X}$ for all $(\mathbf{u}_1, \mathbf{v}_1), (\mathbf{u}_2, \mathbf{v}_2) \in \mathbf{X}$. Similarly, we define the complexification $\mathbf{Y}_\mathbb{C}$ of **Y**.

Under the assumptions (**A0**) and (**A4**), according to lemma 4.1 the embedding $\mathbf{Y}_\mathbb{C} \subset \mathbf{X}_\mathbb{C}$ is dense and continuous. Therefore, there exists a (unique) extension of $\mathbf{X}_\mathbb{C}$, called *extrapolation space* $\mathbf{Y}_\mathbb{C}^*$, which is isometric to the dual space of $\mathbf{Y}_\mathbb{C}$, such that the triple $\mathbf{Y}_\mathbb{C} \subset \mathbf{X}_\mathbb{C} \subset \mathbf{Y}_\mathbb{C}^*$ satisfies the following conditions (see e.g. [36, sectiion 7, chapter 1]):

(1) $\mathbf{Y}_{\mathbb{C}} \subset \mathbf{X}_{\mathbb{C}} \subset \mathbf{Y}_{\mathbb{C}}^*$ with dense and continuous embeddings;
(2) $\{\mathbf{Y}_{\mathbb{C}}^*, \mathbf{Y}_{\mathbb{C}}\}$ forms an adjoint pair with the duality product $\langle\cdot,\cdot\rangle_{\mathbf{Y}_{\mathbb{C}}^*,\mathbf{Y}_{\mathbb{C}}}$;
(3) the duality product $\langle\cdot,\cdot\rangle_{\mathbf{Y}_{\mathbb{C}}^*,\mathbf{Y}_{\mathbb{C}}}$ satisfies

$$\langle(\mathbf{w},\mathbf{z}),(\mathbf{u},\mathbf{v})\rangle_{\mathbf{Y}_{\mathbb{C}}^*,\mathbf{Y}_{\mathbb{C}}} = ((\mathbf{u},\mathbf{v}),(\mathbf{w},\mathbf{z}))_{\mathbf{X}_{\mathbb{C}}} \quad \forall(\mathbf{u},\mathbf{v}) \in \mathbf{Y}_{\mathbb{C}}, \quad \forall(\mathbf{w},\mathbf{z}) \in \mathbf{X}_{\mathbb{C}}.$$

Since the inner-product $(\cdot,\cdot)_{\mathbf{Y}_{\mathbb{C}}}$ is a symmetric sesquilinear form over $\mathbf{Y}_{\mathbb{C}}$, the operator $\mathcal{B}_{\mathbf{Y}_{\mathbb{C}}} : \mathbf{Y}_{\mathbb{C}} \to \mathbf{Y}_{\mathbb{C}}^*$ defined by

$$\langle \mathcal{B}_{\mathbf{Y}_{\mathbb{C}}}(\mathbf{u}_1,\mathbf{v}_1),(\mathbf{u}_2,\mathbf{v}_2)\rangle_{\mathbf{Y}_{\mathbb{C}}^*\times\mathbf{Y}_{\mathbb{C}}} := ((\mathbf{u}_1,\mathbf{v}_1),(\mathbf{u}_2,\mathbf{v}_2))_{\mathbf{Y}_{\mathbb{C}}} \quad \forall\,(\mathbf{u}_1,\mathbf{v}_1),(\mathbf{u}_2,\mathbf{v}_2) \in \mathbf{Y}_{\mathbb{C}}$$

is linear and bounded. In addition, if we define

$$\mathcal{B}(\mathbf{u},\mathbf{v}) := \mathcal{B}_{\mathbf{Y}_{\mathbb{C}}}(\mathbf{u},\mathbf{v}) \quad \forall\,(\mathbf{u},\mathbf{v}) \in D(\mathcal{B}) \tag{4.4}$$

with the domain

$$D(\mathcal{B}) := \{(\mathbf{u},\mathbf{v}) \in \mathbf{Y}_{\mathbb{C}} \mid \mathcal{B}_{\mathbf{Y}_{\mathbb{C}}}(\mathbf{u},\mathbf{v}) \in \mathbf{X}_{\mathbb{C}}\},$$

then $\mathcal{B} : D(\mathcal{B}) \subset \mathbf{X}_{\mathbb{C}} \to \mathbf{X}_{\mathbb{C}}$ is a densely defined and close operator (see [36, theorem 1.25]), and it satisfies many other mathematical properties. Some of them are summarized in the following lemma:

Lemma 4.2 ([36, theorem 2.34 and corollary 2.4]). *Assume that* (**A0**) *and* (**A4**) *hold true. Then, the unbounded operator* $\mathcal{B} : D(\mathcal{B}) \subset \mathbf{X}_{\mathbb{C}} \to \mathbf{X}_{\mathbb{C}}$ *is densely defined, closed, self-adjoint operator and m-accretive. Furthermore, it satisfies*

$$(\mathcal{B}(\mathbf{u}_1,\mathbf{v}_1),(\mathbf{u}_2,\mathbf{v}_2))_{\mathbf{X}_{\mathbb{C}}} = ((\mathbf{u}_1,\mathbf{v}_1),(\mathbf{u}_2,\mathbf{v}_2))_{\mathbf{Y}_{\mathbb{C}}} \quad \forall\ (\mathbf{u}_1,\mathbf{v}_1) \in D(\mathcal{B}),(\mathbf{u}_2,\mathbf{v}_2) \in \mathbf{Y}_{\mathbb{C}}. \tag{4.5}$$

In addition, $(\mathcal{B}, D(\mathcal{B}))$ *is maximal in the sense that if* $(\mathbf{u},\mathbf{v})$ *is an element in* $\mathbf{Y}_{\mathbb{C}}$ *satisfying*

$$((\mathbf{u}^*,\mathbf{v}^*),(\mathbf{e},\mathbf{h}))_{\mathbf{X}_{\mathbb{C}}} = ((\mathbf{u},\mathbf{v}),(\mathbf{e},\mathbf{h}))_{\mathbf{Y}_{\mathbb{C}}} \quad \forall(\mathbf{e},\mathbf{h}) \in \mathbf{Y}_{\mathbb{C}}$$

for some $(\mathbf{u}^*,\mathbf{v}^*) \in \mathbf{X}_{\mathbb{C}}$*, then* $(\mathbf{u},\mathbf{v}) \in D(\mathcal{B})$ *and* $\mathcal{B}(\mathbf{u},\mathbf{v}) = (\mathbf{u}^*,\mathbf{v}^*)$.

From (4.5) we obtain that

$$(\mathcal{B}(\mathbf{u},\mathbf{v}),(\mathbf{u},\mathbf{v}))_{\mathbf{X}_{\mathbb{C}}} = \|(\mathbf{u},\mathbf{v})\|^2_{\mathbf{Y}_{\mathbb{C}}} \geqslant \|(\mathbf{u},\mathbf{v})\|^2_{\mathbf{X}_{\mathbb{C}}} \quad \forall(\mathbf{u},\mathbf{v}) \in D(\mathcal{B}).$$

Then, in view of the compactness of the embedding $D(\mathcal{B}) \subset \mathbf{X}_{\mathbb{C}}$, we infer that there exists a complete orthonormal basis $\{(\mathbf{a}_n,\mathbf{b}_n)\}_{n=1}^{\infty}$ in $\mathbf{X}_{\mathbb{C}}$ such that

$$(\mathcal{B}(\mathbf{u},\mathbf{v}),(\mathbf{u},\mathbf{v}))_{\mathbf{X}_{\mathbb{C}}} = \sum_{n=1}^{\infty} \lambda_n |((\mathbf{u},\mathbf{v}),(\mathbf{a}_n,\mathbf{b}_n))_{\mathbf{X}_{\mathbb{C}}}|^2 \quad \forall\,(\mathbf{u},\mathbf{v}) \in D(\mathcal{B}), \tag{4.6}$$

where $1 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots$, $\lim_{n\to\infty} \lambda_n = +\infty$, and, for every $n \in \mathbb{N}$, $(\mathbf{a}_n,\mathbf{b}_n)$ is the eigenfunction of $\mathcal{B}$ for the eigenvalue of λ_n, i.e.

$$\mathcal{B}(\mathbf{a}_n,\mathbf{b}_n) = \lambda_n(\mathbf{a}_n,\mathbf{b}_n) \quad \forall n \geqslant 1.$$

For every $s \geqslant 0$, the fractional power $\mathcal{B}^s$ of $\mathcal{B}$ can be defined as

$$\mathcal{B}^s(\mathbf{u},\mathbf{v}) := \sum_{n=1}^{\infty} \lambda_n^s((\mathbf{u},\mathbf{v}),(\mathbf{a}_n,\mathbf{b}_n))_{\mathbf{X}_{\mathbb{C}}}(\mathbf{a}_n,\mathbf{b}_n) \quad \forall\,(\mathbf{u},\mathbf{v}) \in D(\mathcal{B}^s), \tag{4.7}$$

where the domain $D(\mathcal{B}^s)$ is given by

$$D(\mathcal{B}^s) = \{(\mathbf{u},\mathbf{v}) \in \mathbf{X}_{\mathbb{C}} \mid \sum_{n=1}^{\infty} \lambda_n^{2s} |((\mathbf{u},\mathbf{v}),(\mathbf{a}_n,\mathbf{b}_n))_{\mathbf{X}_{\mathbb{C}}}|^2 < \infty\}. \tag{4.8}$$

Then, for each $s \geqslant 0$, $\mathcal{B}^s : D(\mathcal{B}^s) \subset \mathbf{X} \to \mathbf{X}$ is also self-adjoint and $D(\mathcal{B}^s)$ is a Banach space equipped with the norm

$$\|(\mathbf{u},\mathbf{v})\|_{D(\mathcal{B}^s)} := \|\mathcal{B}^s(\mathbf{u},\mathbf{v})\|_{\mathbf{X}_{\mathbb{C}}} \quad \forall (\mathbf{u},\mathbf{v}) \in D(\mathcal{B}^s), \tag{4.9}$$

which is also equivalent to the corresponding graph norm of $(\mathcal{B}^s, D(\mathcal{B}^s))$ (for more details, we refer to [30, 35]). Let us mention that

$$D(\mathcal{B}^{1/2}) = \mathbf{Y}_{\mathbb{C}} \tag{4.10}$$

holds with norm equivalence (see [36, theorem 2.33]).

4.2. Verification of VSC and convergence rates

First of all, let us remark that the variational source condition (4.1) is equivalent to

$$\begin{aligned}&((\mathbf{u}^\dagger,\mathbf{v}^\dagger),(\mathbf{u}^\dagger-\mathbf{u},\mathbf{v}^\dagger-\mathbf{v}))_{\mathbf{Y}}\\ &\leqslant \frac{1-\beta}{2}\|(\mathbf{u}^\dagger,\mathbf{v}^\dagger)-(\mathbf{u},\mathbf{v})\|_{\mathbf{Y}}^2 + \Psi(\|S(\mathbf{u}^\dagger,\mathbf{v}^\dagger)-S(\mathbf{u},\mathbf{v})\|_{\mathbf{X}}) \quad \forall\,(\mathbf{u},\mathbf{v}) \in \mathbf{Y}.\end{aligned} \tag{4.11}$$

At this point, we shall recall that $\mathbf{Y}$ only contains real-valued functions. Our goal now is to verify (4.11) for some concave index function $\Psi : (0,\infty) \to (0,\infty)$ and some constant $\beta \in (0,1]$. The arguments used in the following theorem are partly inspired from [5, lemma 5.1] and [21, theorem 2.1].

Theorem 4.3. *Let* **(A0)–(A2)** *and* **(A4)** *be satisfied, and let* $(\mathbf{u}^\dagger,\mathbf{v}^\dagger) \in D(\mathcal{B}^{1/2+s})$ *be a real-valued function with* $s \in (0,1/2]$. *Then, there exists a concave index function* $\Psi : (0,\infty) \to (0,\infty)$ *satisfying the variational source condition (4.11) with* $\beta = \frac{1}{2}$ *and*

$$\Psi(\delta) = O(\delta^{\frac{2s}{2s+1}}) \quad \text{as} \quad \delta \to 0^+. \tag{4.12}$$

Proof. We prove that (4.11) is satisfied for $\beta = 1/2$ and an appropriate index function $\Psi : (0,\infty) \to (0,\infty)$. To this end, let $(\mathbf{u},\mathbf{v}) \in \mathbf{Y}$ and for every $\lambda \geqslant \lambda_1$ we introduce the following orthogonal projection:

$$P_\lambda(\mathbf{w},\mathbf{z}) = \sum_{\lambda_1 \leqslant \lambda_n \leqslant \lambda} ((\mathbf{w},\mathbf{z}),(\mathbf{a}_n,\mathbf{b}_n))_{\mathbf{X}_{\mathbb{C}}}(\mathbf{a}_n,\mathbf{b}_n) \quad \forall\,(\mathbf{w},\mathbf{z}) \in \mathbf{X}_{\mathbb{C}}.$$

We then infer that for all $\lambda \geqslant \lambda_1$,

$$\begin{aligned}&((\mathbf{u}^\dagger,\mathbf{v}^\dagger),(\mathbf{u}^\dagger-\mathbf{u},\mathbf{v}^\dagger-\mathbf{v}))_{\mathbf{Y}}\\ &=\mathrm{Re}\,((I-P_\lambda)(\mathbf{u}^\dagger,\mathbf{v}^\dagger),(\mathbf{u}^\dagger-\mathbf{u},\mathbf{v}^\dagger-\mathbf{v}))_{\mathbf{Y}_{\mathbb{C}}} + \mathrm{Re}\,((\mathbf{u}^\dagger,\mathbf{v}^\dagger),P_\lambda(\mathbf{u}^\dagger-\mathbf{u},\mathbf{v}^\dagger-\mathbf{v}))_{\mathbf{Y}_{\mathbb{C}}}\\ &=: \mathbf{I_1} + \mathbf{I_2},\end{aligned}$$

since $P_\lambda : \mathbf{X}_{\mathbb{C}} \to \mathbf{X}_{\mathbb{C}}$ is self-adjoint.

On the one hand, the Cauchy–Schwarz inequality and Young's inequality yield

$$\begin{aligned}\mathbf{I_1} \leqslant& \|(\mathbf{u}^\dagger - \mathbf{u}, \mathbf{v}^\dagger - \mathbf{v})\|_{\mathbf{Y}} \|(I - P_\lambda)(\mathbf{u}^\dagger, \mathbf{v}^\dagger)\|_{\mathbf{Y}_\mathbb{C}} \\ \leqslant& \frac{1}{4}\|(\mathbf{u}^\dagger - \mathbf{u}, \mathbf{v}^\dagger - \mathbf{v})\|^2_{\mathbf{Y}} + \|(I - P_\lambda)(\mathbf{u}^\dagger, \mathbf{v}^\dagger)\|^2_{\mathbf{Y}_\mathbb{C}}.\end{aligned}$$

From (4.9) and (4.10), it follows that $\|(I - P_\lambda)(\mathbf{u}^\dagger, \mathbf{v}^\dagger)\|^2_{\mathbf{Y}_\mathbb{C}} = \|\mathcal{B}^{1/2}((I - P_\lambda)(\mathbf{u}^\dagger, \mathbf{v}^\dagger))\|^2_{\mathbf{X}_\mathbb{C}}$. Then, the definition (4.7) of $\mathcal{B}^s$ implies

$$\begin{aligned}\|(I - P_\lambda)(\mathbf{u}^\dagger, \mathbf{v}^\dagger)\|^2_{\mathbf{Y}_\mathbb{C}} \leqslant& C\sum_{\lambda_n > \lambda} \lambda_n |((\mathbf{u}, \mathbf{v}), (\mathbf{a}_n, \mathbf{b}_n))_{\mathbf{X}_\mathbb{C}}|^2 \leqslant C\sum_{\lambda_n > \lambda} \frac{\lambda_n^{1+2s}}{\lambda^{2s}} |((\mathbf{u}, \mathbf{v}), (\mathbf{a}_n, \mathbf{b}_n))_{\mathbf{X}_\mathbb{C}}|^2 \\ \leqslant& \frac{C}{\lambda^{2s}}\|\mathcal{B}^{1/2+s}(\mathbf{u}^\dagger, \mathbf{v}^\dagger)\|^2_{\mathbf{X}_\mathbb{C}},\end{aligned}$$

for some constant $C > 0$. In conclusion, we obtain

$$\mathbf{I_1} \leqslant \frac{1}{4}\|(\mathbf{u}^\dagger - \mathbf{u}, \mathbf{v}^\dagger - \mathbf{v})\|^2_{\mathbf{Y}} + \frac{C}{\lambda^{2s}}\|\mathcal{B}^{1/2+s}(\mathbf{u}^\dagger, \mathbf{v}^\dagger)\|^2_{\mathbf{X}_\mathbb{C}}. \tag{4.13}$$

An interplay of proposition 3.3 and the definition of projection P_λ implies that

$$\begin{aligned}\mathbf{I_2} \leqslant& \sum_{\lambda_1 \leqslant \lambda_n \leqslant \lambda} \lambda_n |((\mathbf{u}^\dagger, \mathbf{v}^\dagger), (\mathbf{a}_n, \mathbf{b}_n))_{\mathbf{X}_\mathbb{C}}| \cdot |((\mathbf{u}^\dagger - \mathbf{u}, \mathbf{v}^\dagger - \mathbf{v}), (\mathbf{a}_n, \mathbf{b}_n))_{\mathbf{X}_\mathbb{C}}| \\ \leqslant& \lambda\|(\mathbf{u}^\dagger, \mathbf{v}^\dagger)\|_{\mathbf{X}_\mathbb{C}}\|(\mathbf{u}^\dagger, \mathbf{v}^\dagger) - (\mathbf{u}, \mathbf{v})\|_{\mathbf{X}_\mathbb{C}} \\ =& \lambda\|(\mathbf{u}^\dagger, \mathbf{v}^\dagger)\|_{\mathbf{X}}\|(\mathbf{u}^\dagger, \mathbf{v}^\dagger) - (\mathbf{u}, \mathbf{v})\|_{\mathbf{X}} \\ \leqslant& C_S\lambda\|(\mathbf{u}^\dagger, \mathbf{v}^\dagger)\|_{\mathbf{X}}\|S(\mathbf{u}^\dagger, \mathbf{v}^\dagger) - S(\mathbf{u}, \mathbf{v})\|_{\mathbf{X}} \\ \leqslant& C_S\lambda(\|(\mathbf{u}^\dagger, \mathbf{v}^\dagger)\|_{\mathbf{X}} + 1)\|S(\mathbf{u}^\dagger, \mathbf{v}^\dagger) - S(\mathbf{u}, \mathbf{v})\|_{\mathbf{X}}.\end{aligned} \tag{4.14}$$

Combing (4.13) and (4.14) and noticing that $\lambda \geqslant \lambda_1$ is arbitrary, we have

$$\begin{aligned}&((\mathbf{u}^\dagger, \mathbf{v}^\dagger), (\mathbf{u}^\dagger - \mathbf{u}, \mathbf{v}^\dagger - \mathbf{v}))_{\mathbf{Y}} \\ \leqslant& \frac{1}{4}\|(\mathbf{u}^\dagger - \mathbf{u}, \mathbf{v}^\dagger - \mathbf{v})\|^2_{\mathbf{Y}} \\ &+ \inf_{\lambda \geqslant \lambda_1}\left(\frac{C}{\lambda^{2s}}\|\mathcal{B}^{1/2+s}(\mathbf{u}^\dagger, \mathbf{v}^\dagger)\|^2_{\mathbf{X}_\mathbb{C}} + C_S\lambda(\|(\mathbf{u}^\dagger, \mathbf{v}^\dagger)\|_{\mathbf{X}} + 1)\|S(\mathbf{u}^\dagger, \mathbf{v}^\dagger) - S(\mathbf{u}, \mathbf{v})\|_{\mathbf{X}}\right)\end{aligned}$$

for all $(\mathbf{u}, \mathbf{v}) \in \mathbf{Y}$. Therefore, it remains to show that the function

$$\Psi_s : (0, \infty) \to (0, \infty), \quad \Psi_s(\delta) := \inf_{\lambda \geqslant \lambda_1}\left(\frac{A_s}{\lambda^{2s}} + B_s\lambda\delta\right) \tag{4.15}$$

is a concave index function, where we set constants $A_s := C\|\mathcal{B}^{1/2+s}(\mathbf{u}^\dagger, \mathbf{v}^\dagger)\|^2_{\mathbf{X}_\mathbb{C}}$ and $B_s := C_S(\|(\mathbf{u}^\dagger, \mathbf{v}^\dagger)\|_{\mathbf{X}} + 1)$. As Ψ_s is an infimum of concave functions, we have that $\Psi_s : (0, \infty) \to (0, \infty)$ is concave. In particular, a classical result yields that $\Psi_s : (0, \infty) \to (0, \infty)$ is continuous (see [43, corollary 47.6]). Now, we prove the decay estimate (4.12), which also implies the continuity of Ψ_s at 0. For any $\delta \in (0, \frac{A_s}{B_s\lambda_1^{(2s+1)}}]$, we choose $\lambda = \sqrt[2s+1]{\frac{A_s}{B_s}}\delta^{-\frac{1}{(2s+1)}}$ and obtain that

$$\Psi_s(\delta) \leqslant 2A_s^{\frac{1}{2s+1}} B_s^{\frac{2s}{2s+1}} \delta^{\frac{2s}{(2s+1)}}.$$

Finally, we verify now that Ψ_s is strictly increasing. To this end, let $\delta_1, \delta_2 > 0$ with $\delta_1 < \delta_2$. Since both $\lambda \to \infty$ implies that the right-hand side of (4.15) blows up, the infimum in the definition of $\Psi(\delta_2)$ can be attained at some $\lambda = \lambda^* < +\infty$. Thus it follows that

$$\Psi_s(\delta_1) \leqslant \left(\frac{A_s}{(\lambda^*)^{2s}} + B_s \lambda^* \delta_1 \right) < \Psi_s(\delta_2). \tag{4.16}$$

This completes the proof. □

An interplay of theorem 4.3 and proposition 3.1 yields the following result:

Theorem 4.4. *Assume that* $(\mathbf{A0})$–$(\mathbf{A2})$ *and* $(\mathbf{A4})$ *hold true,* $(\mathbf{u}^\dagger, \mathbf{v}^\dagger) \in D(\mathcal{B}^{1/2+s})$ *with* $1/2 \geqslant s > 0$*, and let* Ψ_s *be the concave index function defined as in (4.15). Furthermore, for every* $\kappa > 0$*, let* $(\mathbf{u}^\delta_\kappa, \mathbf{v}^\delta_\kappa)$ *denote a minimizer of the Tikhonov-regularization problem (2.3).*

(a) If the regularization parameter $\kappa > 0$ *is chosen as* $\kappa = \kappa(\delta) := \frac{2\delta^2}{\Psi_s(\delta)}$*, then*

$$\|(\mathbf{u}^\delta_{\kappa(\delta)}, \mathbf{v}^\delta_{\kappa(\delta)}) - (\mathbf{u}^\dagger, \mathbf{v}^\dagger)\|^2_{\mathbf{Y}} = O(\delta^{\frac{2s}{2s+1}}) \quad \text{as } \delta \to 0^+. \tag{4.17}$$

(b) Let $(\mathbf{A3})$ *be satisfied and let* $(\mathbf{K}^\delta_{\kappa(\delta)}, \mathbf{Q}^\delta_{\kappa(\delta)})$ *be the adjoint state satisfying (3.39) and (3.40) associated with* $(\mathbf{u}^\delta_{\kappa(\delta)}, \mathbf{v}^\delta_{\kappa(\delta)})$*. Then, it holds that*

$$\|(\mathbf{K}^\delta_{\kappa(\delta)}, \mathbf{Q}^\delta_{\kappa(\delta)})\|^2_{C([0,T];\mathbf{X})} = O(\delta^{\frac{2s}{2s+1}}) \quad \text{as } \delta \to 0^+,$$

provided that the regularization parameter $\kappa > 0$ *is chosen as* $\kappa = \kappa(\delta) = \frac{2\delta^2}{\Psi_s(\delta^2)}$*.*

Proof. Assertion (a) is merely a direct consequence of the proposition 3.1 and theorem 4.3. Let $(\mathbf{E}^\delta_{\kappa(\delta)}, \mathbf{H}^\delta_{\kappa(\delta)}) \in C([0,T];\mathbf{X})$ be the mild solution of (2.1) associated with $(\mathbf{u}^\delta_{\kappa(\delta)}, \mathbf{v}^\delta_{\kappa(\delta)})$. To prove the second assertion, we first obtain from (4.17) and the Lipschtiz continuity of operator $S : \mathbf{X} \to \mathbf{X}$ that

$$\|(\mathbf{E}^\delta_{\kappa(\delta)}, \mathbf{H}^\delta_{\kappa(\delta)})(T) - (\mathbf{e}^\dagger, \mathbf{h}^\dagger)\|^2_{\mathbf{X}} = O(\delta^{\frac{2s}{2s+1}}) \quad \text{as } \delta \to 0^+,$$

which, together with (2.2), implies

$$\|(\mathbf{E}^\delta_{\kappa(\delta)}, \mathbf{H}^\delta_{\kappa(\delta)})(T) - (\mathbf{e}^\delta, \mathbf{h}^\delta)\|^2_{\mathbf{X}} = O(\delta^{\frac{2s}{2s+1}}).$$

Then, we use the argument as in the proof of corollary 3.10 to complete the proof. □

4.3. Concrete realization of $D(\mathcal{B}^s)$ by fractional Sobolev spaces

Theorem 4.4 yields a convergence rate result for the Tikhonov regularization method (2.3) under the condition that the true initial value $(\mathbf{u}^\dagger, \mathbf{v}^\dagger)$ belongs to $D(\mathcal{B}^{s+\frac{1}{2}})$, $0 < s \leqslant \frac{1}{2}$. Our goal now is to present an explicit characterization of $D(\mathcal{B}^{s+\frac{1}{2}})$ with the help of fractional Sobolev spaces. To this end, we shall utilize the complex interpolation theory.

If two normed complex Banach space X and Y are continuously embedded in a Hausdorff topological vector space, then for every $\theta \in [0, 1]$, we can define the complex interpolation

$[X, Y]_\theta$ between X and Y. From the classical theory on the complex interpolation space [27, section 2.1] and [36, theorem 2.34 and corollary 2.4], we have the following result.

Lemma 4.5. *Let $\mathcal{B} : D(\mathcal{B}) \subset \mathbf{X}_\mathbb{C} \to \mathbf{X}_\mathbb{C}$ be the self-adjoint operator defined as in lemma 4.2. Then, it holds that*

$$D(\mathcal{B}^s) = [\mathbf{X}_\mathbb{C}, D(\mathcal{B})]_s \quad \forall\, s \in [0, 1],$$

and

$$D(\mathcal{B}^s) = [\mathbf{X}_\mathbb{C}, \mathbf{Y}_\mathbb{C}]_{2s} \quad \forall\, 0 \leqslant s \leqslant \frac{1}{2}$$

with norm equivalence.

In the following, we shall utilize fractional Sobolev spaces of complex valued functions. To more be precise, for any $0 \leqslant s < \infty$, we define

$$H^s(\mathbb{R}^n;\mathbb{C}) := \{u \in \mathcal{S}(\mathbb{R}^n;\mathbb{C})' \mid \|u\|^2_{H^s(\mathbb{R}^n;\mathbb{C})} := \int_{\mathbb{R}^n} (1+|\xi|^2)^s |(\mathcal{F}u)(\xi)|^2 \mathrm{d}\xi < +\infty\},$$

where $\mathcal{F} : \mathcal{S}(\mathbb{R}^n;\mathbb{C})' \to \mathcal{S}(\mathbb{R}^n;\mathbb{C})'$ represents the Fourier transform and $\mathcal{S}(\mathbb{R}^n;\mathbb{C})'$ denotes the tempted distribution space (see, e.g. [35, 36]). Let us point out that $H^s(\mathbb{R}^n;\mathbb{C})$ is a Hilbert space equipped with inner product

$$(u, v)_{H^s(\mathbb{R}^n;\mathbb{C})} := ((1+|\cdot|^2)^{s/2}(\mathcal{F}u)(\cdot), (1+|\cdot|^2)^{s/2}(\mathcal{F}v)(\cdot))_{L^2(\mathbb{R}^n;\mathbb{C})} \quad \forall\, u, v \in H^s(\mathbb{R}^n;\mathbb{C}).$$

In addition, for a bounded domain $\mathcal{O} \subset \mathbb{R}^n$ with a Lipschitz boundary $\partial\mathcal{O}$, the space $H^s(\mathcal{O};\mathbb{C})$ with a possibly non-integer exponent $s \geqslant 0$ is defined as the space of all complex-valued functions $u \in L^2(\mathcal{O};\mathbb{C})$ with some $U \in H^s(\mathbb{R}^n;\mathbb{C})$ such that $U_{|\mathcal{O}} = u$, endowed with the norm

$$\|u\|_{H^s(\mathcal{O};\mathbb{C})} := \inf_{\substack{U_{|\mathcal{O}}=u \\ U \in H^s(\mathbb{R}^n;\mathbb{C})}} \|U\|_{H^s(\mathbb{R}^n;\mathbb{C})}.$$

For every $s \in [0, \infty)$, we denote by $\lfloor s \rfloor \in [0, s]$ the largest integer less or equal to s. In the case of $s \in (0, \infty)$ with $s = \lfloor s \rfloor + \sigma$ and $0 < \sigma < 1$, the norm $\|\cdot\|_{H^s(\mathcal{O};\mathbb{C})}$ is equivalent to (see, e.g. [36, 37]):

$$\left(\|u\|^2_{L^2(\mathcal{O};\mathbb{C})} + \sum_{|\alpha| \leqslant \lfloor s \rfloor} \int\int_{\mathcal{O}\times\mathcal{O}} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2\sigma}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}}.$$

If s is an integer, then $H^s(\mathcal{O};\mathbb{C})$ coincides with the classical Sobolev space. In particular, $H^0(\mathcal{O};\mathbb{C}) = L^2(\mathcal{O};\mathbb{C})$. Moreover, for $s > \frac{1}{2}$, the trace operator $\gamma : H^s(\mathcal{O};\mathbb{C}) \to H^{s-\frac{1}{2}}(\partial\mathcal{O};\mathbb{C})$ is linear and bounded. Let $\mathring{H}^s(\mathcal{O};\mathbb{C})$ denote the closure of $C_0^\infty(\mathcal{O};\mathbb{C})$ with respect to the norm of $H^s(\mathcal{O})$. It is well-known that $C_0^\infty(\mathcal{O};\mathbb{C})$ is dense in $H^s(\mathcal{O};\mathbb{C})$ for all $0 \leqslant s \leqslant 1/2$ (see, e.g. [36, theorem 1.40]), and so $H^s(\mathcal{O};\mathbb{C}) = \mathring{H}^s(\mathcal{O};\mathbb{C})$ for all $0 \leqslant s \leqslant 1/2$. Furthermore, if $\mathcal{O}$ is of class $C^{1,1}$, then

$$\mathring{H}^s(\mathcal{O};\mathbb{C}) = \{u \in H^s(\mathcal{O};\mathbb{C}) \mid \gamma(\frac{\partial^k u}{\partial \nu^k}) = 0 \quad \forall\, 0 \leqslant k \leqslant \lfloor s - 1/2 \rfloor\} \quad \forall\, s \in (\frac{1}{2}, 2] \setminus \{\frac{3}{2}\}$$

(see [17, corollary 1.5.1.6]). Also, in the case of $\mathcal{O}$ being a $C^{1,1}$-domain, it holds that

$$[L^2(\mathcal{O};\mathbb{C}), \mathring{H}^2(\mathcal{O};\mathbb{C})]_s = \begin{cases} \mathring{H}^{2s}(\mathcal{O};\mathbb{C}) & \text{for } 1/4 < s \leqslant 1,\ s \neq \frac{3}{4}, \\ H^{2s}(\mathcal{O};\mathbb{C}) & \text{for } 0 \leqslant s < 1/4, \end{cases} \tag{4.18}$$

with norm equivalence (see [17, theorem 1.5.1.5 and corollary 1.4.4.5] and [27, theorem 11.6]). In the following, we shall make use of (4.18) for $\mathcal{O} = \Omega_j$, $j = 1\ldots, N$. To this aim, we require an additional smoothness assumption on the subdomains Ω_j:

(A5) The domain Ω satisfies **(A4)** with $C^{1,1}$-subdomains Ω_j for all $j = 1, \ldots, N$.

Let us underline that in the case of $N \geqslant 2$, the assumption **(A5)** does not necessarily imply that the whole domain Ω is of class $C^{1,1}$. In fact, we do not require this global $C^{1,1}$-assumption on Ω, since we only apply the formula (4.18) to every subdomain $\mathcal{O} = \Omega_j$ for $j = 1, \ldots, N$.

We introduce next the following function spaces, which will be important in the sequel:

$$\mathcal{X}^s := \{\mathbf{u} \in \mathbf{L}^2(\Omega;\mathbb{C}) \mid \mathbf{u}\,|_{\Omega_j} \in H^{2s}(\Omega_j;\mathbb{C})^3,\, j = 1, 2, \ldots N\},$$
$$\mathring{\mathcal{X}}^s := \{\mathbf{u} \in \mathbf{L}^2(\Omega;\mathbb{C}) \mid \mathbf{u}\,|_{\Omega_j} \in \mathring{H}^{2s}(\Omega_j;\mathbb{C})^3,\ j = 1, 2, \ldots N\},$$

endowed with the norm $\|\mathbf{u}\|_{\mathcal{X}^s} = \left(\sum_{j=1}^{N} \|\mathbf{u}\,|_{\Omega_j}\|^2_{H^{2s}(\Omega_j;\mathbb{C})^3}\right)^{1/2}$. Assuming **(A5)**, it follows from (4.18) that

$$[\mathbf{L}^2(\Omega;\mathbb{C}), \mathring{\mathcal{X}}^1]_s = \begin{cases} \mathring{\mathcal{X}}^s & \text{for } 1/4 < s \leqslant 1, \quad s \neq \frac{3}{4}, \\ \mathcal{X}^s & \text{for } 0 \leqslant s < 1/4. \end{cases} \tag{4.19}$$

Proposition 4.6. *Under* **(A0), (A4)** *and* **(A5)***, we have the following continuous embeddings:*

$$\mathring{\mathcal{X}}^s \times \mathring{\mathcal{X}}^s \subset D(\mathcal{B}^s) \ \text{ for } 1 \geqslant s > \frac{1}{4} \text{ and } s \neq \frac{3}{4},$$

and

$$\mathcal{X}^s \times \mathcal{X}^s \subset D(\mathcal{B}^s) \ \text{ for } 0 \leqslant s < \frac{1}{4}.$$

Proof. Let us define the linear space

$$D(\mathcal{C}) := \mathring{\mathcal{X}}^1 \times \mathring{\mathcal{X}}^1,$$

which is a Banach space under the norm

$$\|(\mathbf{u}, \mathbf{v})\|_{D(\mathcal{C})} = (\|\mathbf{u}\|^2_{\mathcal{X}^1} + \|\mathbf{v}\|^2_{\mathcal{X}^1})^{1/2}. \tag{4.20}$$

Making use of this Banach space, we introduce an unbounded operator $C : D(\mathcal{C}) \subset \mathbf{X}_{\mathbb{C}} \to \mathbf{X}_{\mathbb{C}}$, defined by $\mathcal{C}(\mathbf{u}, \mathbf{v}) = (\mathcal{C}_1\mathbf{u}, \mathcal{C}_2\mathbf{v})$, where

$$\mathcal{C}_1\mathbf{u} = \varepsilon^{-1}\mathbf{u} + \varepsilon^{-1}\mathbf{curl}\, \times \mathbf{curl}\,\mathbf{u} - \nabla\mathrm{div}\,(\varepsilon\mathbf{u}),$$

and

$$\mathcal{C}_2\mathbf{v} = \mu^{-1}\mathbf{v} + \mu^{-1}\mathbf{curl}\, \times \mathbf{curl}\,\mathbf{v} - \nabla\mathrm{div}\,(\mu\mathbf{v}).$$

Let us underline that $\mathcal{C}(\mathbf{u}, \mathbf{v}) \in \mathbf{X}_{\mathbb{C}}$ holds for all $(\mathbf{u}, \mathbf{v}) \in D(\mathcal{C})$ since $\varepsilon\,|_{\Omega_j}, \mu\,|_{\Omega_j} \in C^2$ and $\mathbf{u}\,|_{\Omega_j}, \mathbf{v}\,|_{\Omega_j} \in \mathring{H}^2(\Omega_j;\mathbb{C})^3$ for all $j = 1, 2, \ldots, N$. We recall that every $(\mathbf{u}, \mathbf{v}) \in \mathbf{Y}_{\mathbb{C}}$, it holds

$$\mathrm{Re}(\mathbf{u}, \mathbf{v}), \mathrm{Im}(\mathbf{u}, \mathbf{v}) \in \mathbf{Y} = \{(\mathbf{u}, \mathbf{v}) \in \mathbf{H_0}(\mathbf{curl}) \times \mathbf{H}(\mathbf{curl}) \mid \varepsilon\mathbf{u} \in \mathbf{H}(\mathrm{div}),\ \ \mu\mathbf{v} \in \mathbf{H_0}(\mathrm{div})\}.$$

Therefore, every $(\mathbf{u},\mathbf{v}) \in \mathbf{Y}_{\mathbb{C}}$ satisfies

$$(\mathcal{C}(\mathbf{z}_1,\mathbf{z}_2),(\mathbf{u},\mathbf{v}))_{\mathbf{X}_{\mathbb{C}}} = ((\mathbf{z}_1,\mathbf{z}_2),(\mathbf{u},\mathbf{v}))_{\mathbf{Y}_{\mathbb{C}}} \quad \forall\,(\mathbf{z}_1,\mathbf{z}_2) \in \mathcal{D}_{\mathbb{C}} \times \mathcal{D}_{\mathbb{C}},$$

where

$$\mathcal{D}_{\mathbb{C}} := \{\mathbf{u} \in C_0^\infty(\Omega;\mathbb{C})^3 \;:\; \mathbf{u}\,|_{\Omega_j} \in C_0^\infty(\Omega_j;\mathbb{C})^3 \text{ for } j = 1,2,\cdots,N\}.$$

Then, because $D(\mathcal{C})$ is merely the closure of $\mathcal{D}_{\mathbb{C}} \times \mathcal{D}_{\mathbb{C}}$ under the norm (4.20), it holds that

$$(\mathcal{C}(\mathbf{u}_1,\mathbf{v}_1),(\mathbf{u}_2,\mathbf{v}_2))_{\mathbf{X}_{\mathbb{C}}} = ((\mathbf{u}_1,\mathbf{v}_1),(\mathbf{u}_2,\mathbf{v}_2))_{\mathbf{Y}_{\mathbb{C}}} \quad \forall(\mathbf{u}_1,\mathbf{v}_1) \in D(\mathcal{C}),\;\; (\mathbf{u}_2,\mathbf{v}_2) \in \mathbf{Y}_{\mathbb{C}}.$$

Therefore, it follows from lemma 4.2 that the operator $(\mathcal{C}, D(\mathcal{C}))$ is the restriction of $(\mathcal{B}, D(\mathcal{B}))$ to the domain $D(\mathcal{C})$. Thus, we have that

$$\|(\mathbf{u},\mathbf{v})\|_{D(\mathcal{B})} = \|\mathcal{B}(\mathbf{u},\mathbf{v})\|_{\mathbf{X}_{\mathbb{C}}} = \|\mathcal{C}(\mathbf{u},\mathbf{v})\|_{\mathbf{X}_{\mathbb{C}}} \lesssim \|(\mathbf{u},\mathbf{v})\|_{D(\mathcal{C})} \quad \forall(\mathbf{u},\mathbf{v}) \in D(\mathcal{C}),$$

which ensures that the embedding $D(\mathcal{C}) \subset D(\mathcal{B})$ is continuous. This implies

$$[\mathbf{X}, D(\mathcal{C})]_s \subset [\mathbf{X}, D(\mathcal{B})]_s = D(\mathcal{B}^s) \quad \forall\, s \in [0,1].$$

Now the assertion follows from the above inclusion and (4.19). □

As a consequence of proposition 4.6 and theorem 4.4, we obtain the following result in terms of a fractional Sobolev space instead of an abstract space.

Corollary 4.7. *Assume that* **(A0)–(A2)** *and* **(A4), (A5)** *hold true, and* $(\mathbf{u}^\dagger, \mathbf{v}^\dagger)$ *is real-valued and satisfies*

$$(\mathbf{u}^\dagger, \mathbf{v}^\dagger) \in \mathring{\mathcal{X}}^{s+\frac{1}{2}} \times \mathring{\mathcal{X}}^{s+\frac{1}{2}}$$

with $1/2 \geqslant s > 0$ *and* $s \neq \frac{1}{4}$*. Then, the statement (a) of theorem 4.4 is valid. If we further assume that* **(A3)** *is true, then theorem 4.4 (b) is also valid.*

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References

[1] Ammari H, Chen D and Zou J 2016 Well-posedness of an electric interface model and its finite element approximation *Math. Models Methods Appl. Sci.* **26** 601–25
[2] Anzengruber S W, Hofmann B and Ramlau R 2013 On the interplay of basis smoothness and specific range conditions occurring in sparsity regularization *Inverse Problems* **29** 125002
[3] Anzengruber S W, Hofmann B and Mathé P 2014 Regularization properties of the discrepancy principle for Tikhonov regularization in Banach spaces *Appl. Anal.* **93** 1382–400
[4] Bommer V and Yousept I 2016 Optimal control of the full time-dependent Maxwell equations *ESAIM: Math. Modelling Numer. Anal.* **50** 237–61
[5] Burger M, Flemming J and Hofmann B 2013 Convergence rates in ℓ^1-regularization if the sparsity assumption fails *Inverse Problems* **29** 025013
[6] Bot R I and Hofmann B 2010 An extension of the variational inequality approach for nonlinear ill-posed problems *J. Integral Equ. Appl.* **22** 369–92
[7] Chen D H, Hofmann B and Zou J 2016 Elastic-net regularization versus ℓ^1-regularization for linear inverse problems with quasi-sparse solutions *Inverse Problem* **33** 015004
[8] Engl H W, Hanke M and Neubauer A 1996 *Regularization of Inverse Problems* (*Mathematics and its Applications* vol 375) (Dordrecht: Kluwer)
[9] Engel K J and Nagel R 2000 *One Parameter Semigroups for Linear Evolution Equations* (*Graduate Texts in Mathematics* vol 194) (New York: Springer) (With contributions by S Brendle *et al*)
[10] Engl H W and Zou J 2000 A new approach to convergence rate analysis of Tikhonov regularization for parameter identification in heat conduction *Inverse Problems* **16** 1907–23
[11] Feng H, Jiang D and Zou J 2010 Simultaneous identification of electric permittivity and magnetic permeability *Inverse Problems* **26** 095009
[12] Feng H, Jiang D and Zou J 2012 Convergence rates of Tikhonov regularizations for parameter identification in a parabolic-elliptic system *Inverse Problems* **28** 104002
[13] Flemming J 2010 Theory and examples of variational regularization with non-metric fitting functionals *J. Inverse Ill-Posed Problems* **18** 677–99
[14] Flemming J 2012 *Generalized Tikhonov Regularization and Modern Convergence Rate Theory in Banach Spaces* (Aachen: Shaker Verlag)
[15] Girault V and Raviart P A 2012 *Finite Element Methods for Navier–Stokes Equations: theory and Algorithms* vol 5 (New York: Springer)
[16] Grasmair M 2010 Generalized Bregman distances and convergence rates for non-convex regularization methods *Inverse Problems* **26** 115014
[17] Grisvard P 2011 *Elliptic Problems in Nonsmooth Domains* (Philadelphia, PA: SIAM)
[18] Hofmann B, Kaltenbacher B, Pöschl C and Scherzer O 2007 A convergence rates result for Tikhonov regularization in Banach spaces with non-smooth operators *Inverse Problems* **23** 987–1010
[19] Hohage T and Weidling F 2015 Verification of a variational source condition for acoustic inverse medium scattering problems *Inverse Problem* **31** 075006
[20] Hohage T and Weidling F 2017 Variational source condition and stability estimates for inverse electromagnetic medium scattering problems *Inverse Problems Imaging* **11** 203–20
[21] Hohage T and Weidling F 2017 Characerizations of variational source conditions, converse results, and maxisets of spectral regularization methods *SIAM J. Numer. Anal.* **55** 598–620
[22] Hofmann B and Mathé P 2012 Parameter choice in Banach space regularization under variational inequalities *Inverse Problems* **28** 104006
[23] Lechleiter A and Rieder A 2008 Newton regularizations for impedance tomography: convergence by local injectivity *Inverse Problems* **24** 065009
[24] Kaltenbacher B, Kaltenbacher M and Reitzinger S 2003 Identification of nonlinear *B-H* curves based on magnetic field computations and multigrid methods for ill-posed problems *Eur. J. Appl. Math.* **14** 15–38
[25] Kaltenbacher B, Schöpfer F and Schuster T 2009 Iterative methods for nonlinear ill-posed problems in Banach spaces: convergence and applications to parameter identification problems *Inverse Problems* **25** 065003
[26] Kirsch A and Rieder A 2016 Inverse problems for abstract evolution equations with applications in electrodynamics and elasticity *Inverse Problems* **32** 085001
[27] Lions J L and Magenes E 2012 *Non-homogeneous Boundary Value Problems and Applications* (New York: Springer)

[28] Monk P 2003 *Finite Element Methods for Maxwell's Equations* (Oxford: Clarendon)
[29] Picard R 1984 An elementary proof for a compact imbedding result in generalized electromagnetic theory *Math. Z.* **187** 151–64
[30] Reed M and Simon S 1980 *Methods of Modern Mathematical Physics* (*Functional Analysis* vol 1) (New York: Academic)
[31] Seidman T I and Vogel C R 1989 Well posedness and convergence of some regularisation methods for non-linear ill posed problems *Inverse Problems* **5** 227
[32] Schuster T, Kaltenbacher B, Hofmann B and Kazimierski K S 2012 *Regularization Methods in Banach Spaces* (*Radon Series on Computational and Applied Mathematics* vol 10) (Berlin: Walter de Gruyter)
[33] Pazy A 1983 *Semigroups of Linear Operators with Applications to Partial Differential Equations* (New York: Springer)
[34] Weck N 1974 Maxwell's boundary value problem on Riemannian manifolds with nonsmooth boundaries *J. Math. Anal. Appl.* **46** 410–37
[35] Wloka J 1987 *Partial Differential Equations* (Cambridge: Cambridge University Press)
[36] Yagi A 2009 *Abstract Parabolic Evolution Equations and their Applications* (New York: Springer)
[37] Yagi A 2008 H^∞-functional calculus and characterization of domains of fractional powers *Proc. Workshop on 'Operator Theory and Applications'* (*Operator Theory: Advances and Applications* vol 187) ed R Curto *et al* (Basel: Birkhäuser) pp 217–35
[38] Yousept I 2012 Optimal control of Maxwell's equations with regularized state constraints *Comput. Optim. Appl.* **52** 559–81
[39] Yousept I 2013 Optimal control of quasilinear $\boldsymbol{H}(\mathbf{curl})$-elliptic partial differential equations in magnetostatic field problems *SIAM J. Control Optim.* **51** 3624–51
[40] Yousept I 2017 Optimal control of non-smooth hyperbolic evolution Maxwell equations in type-II superconductivity *SIAM J. Control Optim.* **55** 2305–32
[41] Yousept I 2017 Hyperbolic Maxwell variational inequalities for Bean's critical-state model in type-II superconductivity *SIAM J. Numer. Anal.* **55** 2444–64
[42] Yousept I 2018 Hyperbolic Maxwell variational inequalities of the second kind preprint SPP1962-069 (https://spp1962.wias-berlin.de/preprints.php)
[43] Zeidler E 2013 *Nonlinear Functional Analysis and its Applications: III: Variational Methods and Optimization* (New York: Springer)