SIAM J. NUMER. ANAL.
Vol. 58, No. 3, pp. 1941–1964

ADAPTIVE EDGE ELEMENT APPROXIMATION FOR H(CURL) ELLIPTIC VARIATIONAL INEQUALITIES OF SECOND KIND*

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Abstract. This paper is concerned with the analysis of an adaptive edge element method for solving elliptic **curl-curl** variational inequalities of second kind. We derive a posteriori error estimators based on a special combination of the Moreau–Yosida regularization and Nédélec's edge elements of first family. With the help of these a posteriori error estimators, an AFEM algorithm is proposed and studied. We are able to establish both the reliability and the efficiency of these estimators, by means of a special linear auxiliary problem involving the discrete Moreau–Yosida-regularized dual formulation, along with a local regular decomposition for **H**(**curl**)-functions and the bubble functions. Furthermore, we demonstrate the strong convergence of the sequence of the edge element solutions generated by the adaptive algorithm toward the solution of a limiting problem, by first achieving the convergence of the maximal error indicator and the residual corresponding to the sequence of the adaptive edge element solutions, under a reasonable condition on the regularization parameter in terms of the adaptive mesh size. Three-dimensional numerical experiments are presented to verify the robustness and effectiveness of the adaptive algorithm when it is applied to a problem arising from the type-II (high-temperature) superconductivity.

Key words. curl-curl elliptic variational inequalities, a posteriori error analysis, Moreau–Yosida regularization, edge elements, convergence analysis, superconductivity

AMS subject classifications. 78M30, 78M10, 81J05

DOI. 10.1137/19M1281320

1. Introduction.

The adaptive finite element method (AFEM) based on a posteriori error estimators is a useful technique to increase the numerical accuracy of solutions to PDE problems in certain sensitive regions of the concerned domain. For variational inequalities (VIs) of the first kind or obstacle problems, these regions are typically the (a priori unknown) free boundaries which correspond to the interfaces between the active and inactive areas of the obstacle. AFEM consists of a repeating execution of the loop:

$$\text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{MARK} \rightarrow \text{REFINE}.$$

Typically, when dealing with obstacle problems, AFEM refines the mesh adaptively where the solution is close to the obstacle, whereas the mesh on the remaining domain stays relatively coarse. This procedure yields a practically important feature to predict and detect the free boundaries without a priori knowledge. For $H^1(\Omega)$-elliptic obstacle problems, earlier results go back to [23, 27], and wide studies can be found in

*Received by the editors August 14, 2019; accepted for publication (in revised form) April 8, 2020; published electronically June 23, 2020.
https://doi.org/10.1137/19M1281320
Funding: The work of the first and second authors was supported by the German Research Foundation (DFG) Priority Program DFG SPP 1962 "Non-Smooth and Complementarity-Based Distributed Parameter Systems: Simulation and Hierarchical Optimization," project YO 159/2-2. The work of the third author was supported by the Hong Kong RGC General Research fund grants 14306718 and 14304517 and by the National Natural Science Foundation of China/Hong Kong RGC Joint Research Scheme 2016/17 project N_CUHK437/16.
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this direction; see [9, 12, 14, 33] and the references therein. However, many important physical phenomena, including Bingham fluid, friction, and high-temperature superconductivity (HTS), cannot be modeled by obstacle problems, hence leading to VIs of second kind. Bostan, Han, and Reddy [7] were the first to propose a duality approach to derive reliable a posteriori error estimators for $H^1(\Omega)$-elliptic VIs of second kind. Some years later, Wang and Han [39] adapted the idea by Braess [8] to prove the efficiency of the proposed estimators by considering an auxiliary linear equation taking the associated discrete dual variable into account.

The first contribution toward residual-type a posteriori error estimators for edge element methods in $\mathbf{H}(\mathbf{curl})$-elliptic equations go back to Beck et al. [6]. Schöberl [35] established some stability estimates for a Clément-type quasi-interpolation operator, which turned out to be a very useful tool in the a posteriori error analysis of Maxwell's equations. The strong convergence of AFEM algorithms for various Maxwell-type equations with edge elements was analyzed in [10, 11, 13, 15, 24, 41]. Some of these developments relied on a key strategy with limiting spaces, which was initially adopted by Babuška and Vogelius [3] for a one-dimensional boundary value problem and then extended to several higher-dimensional problems by Morin, Siebert, and Veeser [31]. We refer the reader to [43, 46] for edge element methods for optimal control problems.

To the best of the authors' knowledge, a posteriori error analysis and adaptive edge element methods for elliptic **curl-curl** VIs still remain an open research area (see [42] for recent mathematical results on (full) hyperbolic Maxwell VIs). In particular, such a class of problems works with the space $\mathbf{H}(\mathbf{curl})$ (instead of H^1) and features various special singularities [16, 17, 18]. These facts, along with the main difficulties arising from the VI character, make the numerical analysis, especially the rigorous a posteriori error analysis of the resulting AFEM, rather challenging.

In this work, we propose and analyze a posteriori error estimators and an AFEM algorithm for $\mathbf{H}(\mathbf{curl})$-elliptic VIs of second kind. Due to the **curl-curl** structure involved, our a posteriori error estimators require a local divergence regularity property of the dual variable. For this reason, unlike all the aforementioned contributions, we make use of a special combination of the Moreau–Yosida regularization and Nédélec's edge elements of first family [32]. We are able to demonstrate that the proposed error estimators are both reliable and efficient. More important, under a certain condition on the regularization parameter depending on the adaptive mesh, we can even establish the strong convergence of the AFEM algorithm. Let us point out that the Moreau–Yosida regularization is a key feature which is not only crucial to our theoretical analysis but also brings a significant advantage to the numerical implementation of our new AFEM. In fact, it makes our implementation much more realistic and efficient. Our numerical realizations are carried out by means of the efficient and robust semismooth Newton method [25] that demands a certain regularity of the dual formulation. This regularity property is well satisfied by the Moreau–Yosida approximation but in general not by the original VI of second kind (cf. [19]). This work appears to be the first contribution that makes full use of the Moreau–Yosida regularization with its great flexibility and advantage in the a posteriori error analysis. The regularization strategy is very different from the finite element discretization of the VIs as was done in the previous studies [7, 39] for $H^1(\Omega)$-elliptic VIs of second kind.

We end this section by introducing the $\mathbf{H}(\mathbf{curl})$-elliptic VIs of second kind of our interest and outlining the main results of this work. We consider the VI of the following form: Find $\mathbf{E} \in \mathbf{H}_0(\mathbf{curl})$ such that

$$\text{(VI)} \qquad a(\mathbf{E}, \mathbf{v} - \mathbf{E}) + \varphi(\mathbf{v}) - \varphi(\mathbf{E}) \geq \int_\Omega \mathbf{f} \cdot (\mathbf{v} - \mathbf{E}) \, d\boldsymbol{x} \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}),$$

where $a\colon \mathbf{H}_0(\mathbf{curl}) \times \mathbf{H}_0(\mathbf{curl}) \to \mathbb{R}$ is a bilinear form defined by

$$a(\mathbf{v}, \mathbf{w}) = \int_\Omega \epsilon \mathbf{v} \cdot \mathbf{w} \, d\boldsymbol{x} + \int_\Omega \mu^{-1} \, \mathbf{curl}\, \mathbf{v} \cdot \mathbf{curl}\, \mathbf{w} \, d\boldsymbol{x}$$

and $\varphi\colon \mathbf{L}^2(\Omega) \to \mathbb{R}$ is a nonlinear and nonsmooth functional of the form

$$\varphi(\mathbf{v}) = \int_\Omega j_c(\boldsymbol{x}) |\mathbf{v}(\boldsymbol{x})| \, d\boldsymbol{x}. \tag{1.1}$$

The specific assumptions on ϵ, μ, j_c and $\mathbf{f}$ are stated in Assumption 2.1. The remainder of this work is structured as follows. First, we introduce the Moreau–Yosida regularization for the dual formulation of (VI) (see (3.3)). After showing a crucial regularity property for the dual variable of the regularized problem, we propose the a posteriori error estimator (3.17). Thereafter, its reliability is proven in Theorem 3.6 by considering the linear auxiliary problem (3.5) and using the Schöberl local regular decomposition (cf. Lemma 3.5). The efficiency of the estimators, stated in Theorem 3.7, follows from a standard argumentation with bubble functions (cf. [1]). With the help of these essential properties, we present the adaptive edge element algorithm for (VI). The main result is a strong convergence theorem (see Theorem 4.10) of the sequence of adaptive solutions generated by Algorithm 4.1 toward the unique solution of (VI). Therefore, the limiting space (4.6) as well as the corresponding limiting variational inequality (VI_∞) are the starting points for all that follows. First, the strong convergence toward this limiting problem is established. Hereafter, under a specific condition on the regularization parameter depending on the adaptive mesh (Assumption 4.6), we derive convergence results for the maximal error indicator and the residual corresponding to the sequence of adaptive solutions (Lemmas 4.7 and 4.8). By means of these convergence properties, we are able to prove that the solution to the limiting problem (VI_∞) coincides with the one to (VI). Hence, strong convergence of Algorithm 4.1 follows as an immediate consequence. We conclude the work by presenting numerical results for an important physical application in the type-II (high-temperature) superconductivity [40, 44, 45].

2. Preliminaries.

We consider a bounded, polyhedral and simply connected domain $\Omega \subset \mathbb{R}^3$ with a connected Lipschitz boundary. For a given Banach space X, we denote its norm by $\|\cdot\|_X$ and the duality pairing with the corresponding dual space X^* by $\langle \cdot, \cdot \rangle$. If X is a Hilbert space, then $(\cdot, \cdot)_X$ stands for its scalar product and $\|\cdot\|_X$ for the induced norm. In the case of $X = \mathbb{R}^n$, we renounce the subscript in the (Euclidean) norm and write $|\cdot|$. The Euclidean scalar product is denoted by a dot. In the case that $X = \mathbf{L}^2(\omega)$ for some $\omega \subset \Omega$, its norm is denoted by $\|\cdot\|_{0,\omega}$. Hereinafter a bold type always indicates a vector-valued function or a vector-valued space. Now let us introduce the most basic Hilbert spaces that will be used throughout this work:

$$\mathbf{H}(\mathbf{curl}) \coloneqq \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \mathbf{curl}\, \mathbf{v} \in \mathbf{L}^2(\Omega)\} \quad \text{and} \quad \mathbf{H}(\mathrm{div}) \coloneqq \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \mathrm{div}\, \mathbf{v} \in \mathbf{L}^2(\Omega)\},$$

where $\mathbf{curl}$ and div are understood in the distributional sense. As usual, $\mathbf{C}_0^\infty(\Omega)$ denotes the space of all infinitely differentiable functions with compact support in Ω.

The space $\mathbf{H}_0(\mathbf{curl})$ stands for the closure of $\mathbf{C}_0^\infty(\Omega)$ with respect to the $\mathbf{H}(\mathbf{curl})$-norm.

Next, we present all the necessary assumptions for the material parameters and the given data in (VI).

Assumption 2.1 (material parameters and given data).

(A1) There are polyhedral Lipschitz subdomains Ω_j in Ω, $j = 1, \ldots, M$, such that

$$\Omega_i \cap \Omega_j = \emptyset \quad \text{for } i \neq j \text{ and } \quad \bar{\Omega} = \bigcup_{j=1}^{M} \bar{\Omega}_j.$$

Furthermore, the material parameters ϵ, μ, j_c satisfy

$$\epsilon(\boldsymbol{x}) = c_i^\epsilon \quad \text{and} \quad \mu(\boldsymbol{x}) = c_i^\mu \quad \text{and} \quad j_c(\boldsymbol{x}) = c_i^{j_c} \quad \forall \boldsymbol{x} \in \Omega_i,\, i \in \{1, \ldots, M\}$$

for positive constants $c_i^\epsilon, c_i^\mu > 0$ and a nonnegative constant $c_i^{j_c} \geq 0$ for all $i \in \{1, \ldots, M\}$.

(A2) The source $\mathbf{f}$ of (VI) lies in $\mathbf{L}^2(\Omega)$ and satisfies the divergence-free condition:

$$(\mathbf{f}, \nabla\phi)_{0,\Omega} = 0 \quad \forall \phi \in H_0^1(\Omega).$$

Under Assumption 2.1 the bilinear form a is continuous and coercive; i.e., there are positive constants $0 < \underline{\kappa} < \overline{\kappa}$ depending only on ϵ and μ such that

$$|a(\mathbf{v}, \mathbf{w})| \leq \overline{\kappa} \|\mathbf{v}\|_{\mathbf{H}(\mathbf{curl})} \|\mathbf{w}\|_{\mathbf{H}(\mathbf{curl})} \quad \forall \mathbf{v}, \mathbf{w} \in \mathbf{H}_0(\mathbf{curl}), \tag{2.1}$$

$$a(\mathbf{v}, \mathbf{v}) \geq \underline{\kappa} \|\mathbf{v}\|^2_{\mathbf{H}(\mathbf{curl})} \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}). \tag{2.2}$$

Due to (2.1), (2.2), and its symmetry, the bilinear form a defines a scalar product whose induced norm $\|\cdot\|_a := \sqrt{a(\cdot,\cdot)}$ is equivalent to $\|\cdot\|_{\mathbf{H}(\mathbf{curl})}$. Furthermore, the induced norm over an arbitrary measurable set $\omega \subset \Omega$ is denoted by $\|\cdot\|_{a,\omega}$.

We close this section by introducing the discrete approximation to (VI). Let $\mathcal{T}_0$ be a shape-regular triangulation of $\bar{\Omega}$ such that ϵ, μ, and j_c are constant in every $T \in \mathcal{T}_0$, and let $\mathbb{T}$ be the set of all possible conforming triangulations obtained from $\mathcal{T}_0$ by successive bisection. One key property of the refinement process ensures that all constants depending only on the shape regularity of any $\mathcal{T} \in \mathbb{T}$ are uniformly bounded by a constant depending only on the initial mesh $\mathcal{T}_0$ (cf. [37]). For any $\mathcal{T} \in \mathbb{T}$ the finite element space of Nédélec's first family of edge elements is defined by

$$\mathbf{V}_\mathcal{T} := \{\mathbf{v}_\mathcal{T} \in \mathbf{H}_0(\mathbf{curl}) \,:\, \mathbf{v}_\mathcal{T}|_T = \mathbf{a}_T + \mathbf{b}_T \times \boldsymbol{x} \text{ with } \mathbf{a}_T, \mathbf{b}_T \in \mathbb{R}^3,\ \forall T \in \mathcal{T}\}.$$

We are now ready to formulate the edge element approximation to (VI).

Find $\mathbf{E}_\mathcal{T} \in \mathbf{V}_\mathcal{T}$ such that

$$a(\mathbf{E}_\mathcal{T}, \mathbf{v}_\mathcal{T} - \mathbf{E}_\mathcal{T}) + \varphi(\mathbf{v}_\mathcal{T}) - \varphi(\mathbf{E}_\mathcal{T}) \geq \int_\Omega \mathbf{f} \cdot (\mathbf{v}_\mathcal{T} - \mathbf{E}_\mathcal{T})\, d\boldsymbol{x} \quad \forall \mathbf{v}_\mathcal{T} \in \mathbf{V}_\mathcal{T}. \tag{$\mathrm{VI}_\mathcal{T}$}$$

Existence and uniqueness of the solutions to (VI) and ($\mathrm{VI}_\mathcal{T}$) follow by the classical theory of elliptic VIs (cf. [28, Theorem 2.2]).

3. A posteriori error analysis. As already pointed out in the introduction, our adaptive algorithm is based on efficient and reliable a posteriori error estimators. In order to establish this, we introduce some additional notation: By $\mathcal{F}_\mathcal{T}$, we denote the set of all faces in $\mathcal{T} \in \mathbb{T}$, and $\mathcal{F}_\mathcal{T}(\Omega)$ stands for the set of all interior faces. Let $h_T = \mathrm{diam}(T)$ for $T \in \mathcal{T}$ and $h_F = \mathrm{diam}(F)$ for $F \in \mathcal{F}_\mathcal{T}$. Furthermore, we use D_T (resp., D_F) to denote the union of all elements that have a nonempty intersection with $T \in \mathcal{T}$ (resp., $F \in \mathcal{F}_\mathcal{T}$). Finally, for $T \in \mathcal{T}$, we define the patch set ω_T as the union of all elements sharing a common face with T, and for any face $F \in \mathcal{F}_\mathcal{T}$ shared by two elements $K, \tilde{K} \in \mathcal{T}$, we set $\omega_F = K \cup \tilde{K}$.

A classical result from the theory of VIs yields the existence of a Lagrange-multiplier for (VI) (cf. [22]). Let $\mathbf{E} \in \mathbf{H}_0(\mathbf{curl})$ be the unique solution to (VI). Then there exists a unique $\boldsymbol{\lambda} \in \mathbf{L}^\infty(\Omega)$ such that

$$(3.1)\qquad \begin{cases} a(\mathbf{E}, \mathbf{v}) + \int_\Omega \boldsymbol{\lambda} \cdot \mathbf{v}\, d\boldsymbol{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\boldsymbol{x} \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}) \\ |\boldsymbol{\lambda}(\boldsymbol{x})| \le j_c(\boldsymbol{x}), \quad \boldsymbol{\lambda}(\boldsymbol{x}) \cdot \mathbf{E}(\boldsymbol{x}) = j_c(\boldsymbol{x})|\mathbf{E}(\boldsymbol{x})| \text{ for a.e. } \boldsymbol{x} \in \Omega. \end{cases}$$

For (3.1), we denote the active and inactive sets by

$$\mathcal{A} \coloneqq \{\boldsymbol{x} \in \Omega \,:\, |\mathbf{E}(\boldsymbol{x})| > 0\} \quad \text{and} \quad \mathcal{I} = \Omega \backslash \mathcal{A}.$$

Next, we introduce the Moreau–Yosida regularization $\psi_\gamma \colon \mathbb{R}^3 \to \mathbb{R}$ of $|\cdot|$ by

$$\psi_\gamma(\boldsymbol{x}) \coloneqq \begin{cases} |\boldsymbol{x}| - \dfrac{1}{2\gamma} & \text{for } |\boldsymbol{x}| \ge \dfrac{1}{\gamma}, \\ \dfrac{\gamma}{2}|\boldsymbol{x}|^2 & \text{for } |\boldsymbol{x}| < \dfrac{1}{\gamma} \end{cases}$$

and consider the regularized version of (VI$_\mathcal{T}$): Find $\mathbf{E}^\gamma_\mathcal{T} \in \mathbf{V}_\mathcal{T}$ such that

$$(3.2)\qquad a(\mathbf{E}^\gamma_\mathcal{T}, \mathbf{v}_\mathcal{T} - \mathbf{E}^\gamma_\mathcal{T}) + \varphi_\gamma(\mathbf{v}_\mathcal{T}) - \varphi_\gamma(\mathbf{E}^\gamma_\mathcal{T}) \ge \int_\Omega \mathbf{f} \cdot (\mathbf{v}_\mathcal{T} - \mathbf{E}^\gamma_\mathcal{T})\, d\boldsymbol{x} \quad \forall \mathbf{v}_\mathcal{T} \in \mathbf{V}_\mathcal{T},$$

where $\varphi_\gamma(\mathbf{v}) \coloneqq \int_\Omega j_c(\boldsymbol{x})\psi_\gamma(\mathbf{v}(\boldsymbol{x}))\, d\boldsymbol{x}$ for $\mathbf{v} \in \mathbf{L}^2(\Omega)$ and $\gamma > 0$. The next lemma states some helpful properties of the Moreau–Yosida regularization (see [34, Lemma 5.17]).

LEMMA 3.1. *Let $\{\mathbf{v}_\gamma\}_{\gamma>0} \subset \mathbf{L}^2(\Omega)$ and $\mathbf{v} \in \mathbf{L}^2(\Omega)$. For every $\gamma > 0$ it holds that $\varphi_\gamma(\mathbf{v}) \le \varphi(\mathbf{v})$ and the following convergence properties are satisfied:*

$$\mathbf{v}_\gamma \rightharpoonup \mathbf{v} \text{ weakly in } \mathbf{L}^2(\Omega) \quad \Rightarrow \quad \liminf_{\gamma\to\infty} \varphi_\gamma(\mathbf{v}_\gamma) \ge \varphi(\mathbf{v}),$$
$$\mathbf{v}_\gamma \to \mathbf{v} \text{ strongly in } \mathbf{L}^2(\Omega) \quad \Rightarrow \quad \limsup_{\gamma\to\infty} \varphi_\gamma(\mathbf{v}_\gamma) \le \varphi(\mathbf{v}).$$

Moreover, $\varphi_\gamma \colon \mathbf{L}^2(\Omega) \to \mathbb{R}$ is Gâteaux-differentiable for every $\gamma > 0$.

Thanks to the Gâteaux-differentiability of φ_γ, (3.2) is equivalent to finding $\mathbf{E}^\gamma_\mathcal{T} \in \mathbf{V}_\mathcal{T}$ such that

$$(3.3)\qquad \begin{cases} a(\mathbf{E}^\gamma_\mathcal{T}, \mathbf{v}_\mathcal{T}) + \int_\Omega \boldsymbol{\lambda}^\gamma_\mathcal{T} \cdot \mathbf{v}_\mathcal{T}\, d\boldsymbol{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v}_\mathcal{T}\, d\boldsymbol{x} \quad \forall \mathbf{v}_\mathcal{T} \in \mathbf{V}_\mathcal{T} \\ \boldsymbol{\lambda}^\gamma_\mathcal{T}(\boldsymbol{x}) = j_c(\boldsymbol{x}) \dfrac{\gamma \mathbf{E}^\gamma_\mathcal{T}(\boldsymbol{x})}{\max\{1, \gamma|\mathbf{E}^\gamma_\mathcal{T}(\boldsymbol{x})|\}} \quad \text{for a.e. } \boldsymbol{x} \in \Omega. \end{cases}$$

In this context, the active and inactive sets are given by

$$\mathcal{A}_\gamma := \{\boldsymbol{x} \in \Omega \,:\, \gamma|\mathbf{E}^\gamma_\mathcal{T}(\boldsymbol{x})|>1\} \quad \text{and} \quad \mathcal{I}_\gamma := \Omega\backslash\mathcal{A}_\gamma. \tag{3.4}$$

Since $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^\gamma_\mathcal{T}$ are essentially bounded in Ω, we may interpret them as elements in $\mathbf{H}_0(\mathbf{curl})^*$ with the operator norm

$$\|\boldsymbol{\lambda}\|_{*,a} := \sup\left\{\int_\Omega \boldsymbol{\lambda}\cdot\mathbf{v}\,d\boldsymbol{x} \,:\, \mathbf{v}\in\mathbf{H}_0(\mathbf{curl}),\ \|\mathbf{v}\|_a = 1\right\}.$$

As a starting point for a posteriori error analysis, we consider the auxiliary problem

$$a(\mathbf{z},\mathbf{v}) + \int_\Omega \boldsymbol{\lambda}^\gamma_\mathcal{T}\cdot\mathbf{v}\,d\boldsymbol{x} = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,d\boldsymbol{x} \quad \forall\mathbf{v}\in\mathbf{H}_0(\mathbf{curl}), \tag{3.5}$$

which admits a unique solution $\mathbf{z}\in\mathbf{H}_0(\mathbf{curl})$.

LEMMA 3.2. *Under Assumption* 2.1 *and for* $C = \max\{5, 6\|j_c\|_{L^1(\Omega)}\}$, *there holds that*

$$\|\mathbf{E}^\gamma_\mathcal{T} - \mathbf{E}\|_a^2 + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^\gamma_\mathcal{T}\|_{*,a}^2 \le C\left(\|\mathbf{E}^\gamma_\mathcal{T} - \mathbf{z}\|_a^2 + \frac{1}{\gamma}\right).$$

Proof. We begin by subtracting (3.1) from (3.5) to obtain

$$a(\mathbf{z}-\mathbf{E},\mathbf{v}) = \int_\Omega (\boldsymbol{\lambda} - \boldsymbol{\lambda}^\gamma_\mathcal{T})\cdot\mathbf{v}\,d\boldsymbol{x} \quad \forall\mathbf{v}\in\mathbf{H}_0(\mathbf{curl}). \tag{3.6}$$

Therefore,

$$a(\mathbf{E}^\gamma_\mathcal{T}-\mathbf{E},\mathbf{v}) = a(\mathbf{E}^\gamma_\mathcal{T}-\mathbf{z},\mathbf{v}) + \int_\Omega (\boldsymbol{\lambda} - \boldsymbol{\lambda}^\gamma_\mathcal{T})\cdot\mathbf{v}\,d\boldsymbol{x} \quad \forall\mathbf{v}\in\mathbf{H}_0(\mathbf{curl}). \tag{3.7}$$

Next, we exploit properties of $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^\gamma_\mathcal{T}$ in (3.1) and (3.3) and prove that

$$\int_\Omega (\boldsymbol{\lambda} - \boldsymbol{\lambda}^\gamma_\mathcal{T})\cdot(\mathbf{E}^\gamma_\mathcal{T}-\mathbf{E})\,d\boldsymbol{x} \le \frac{1}{\gamma}\|j_c\|_{L^1(\Omega)}. \tag{3.8}$$

To this aim, we divide Ω into $\mathcal{A}\cap\mathcal{A}_\gamma, \mathcal{A}\cap\mathcal{I}_\gamma, \mathcal{I}\cap\mathcal{A}_\gamma$ as well as $\mathcal{I}\cap\mathcal{I}_\gamma$ and show pointwise estimates for the integrand in (3.8). For $\boldsymbol{x}\in\mathcal{A}\cap\mathcal{A}_\gamma$, (3.1) and (3.3) imply

$$\begin{aligned}
&(\boldsymbol{\lambda}(\boldsymbol{x}) - \boldsymbol{\lambda}^\gamma_\mathcal{T}(\boldsymbol{x}))\cdot(\mathbf{E}^\gamma_\mathcal{T}(\boldsymbol{x}) - \mathbf{E}(\boldsymbol{x}))\\
&\quad= \boldsymbol{\lambda}(\boldsymbol{x})\cdot\mathbf{E}^\gamma_\mathcal{T}(\boldsymbol{x}) - \boldsymbol{\lambda}(\boldsymbol{x})\cdot\mathbf{E}(\boldsymbol{x}) + \boldsymbol{\lambda}^\gamma_\mathcal{T}(\boldsymbol{x})\cdot\mathbf{E}(\boldsymbol{x}) - \boldsymbol{\lambda}^\gamma_\mathcal{T}(\boldsymbol{x})\cdot\mathbf{E}^\gamma_\mathcal{T}(\boldsymbol{x})\\
&\quad\le j_c(\boldsymbol{x})|\mathbf{E}^\gamma_\mathcal{T}(\boldsymbol{x})| - j_c(\boldsymbol{x})|\mathbf{E}(\boldsymbol{x})| + j_c(\boldsymbol{x})|\mathbf{E}(\boldsymbol{x})| - j_c(\boldsymbol{x})|\mathbf{E}^\gamma_\mathcal{T}(\boldsymbol{x})| = 0.
\end{aligned}$$

For $\boldsymbol{x}\in\mathcal{A}\cap\mathcal{I}_\gamma$, (3.1) and (3.3) yield $j_c(\boldsymbol{x})\mathbf{E}^\gamma_\mathcal{T}(\boldsymbol{x}) = \gamma^{-1}\boldsymbol{\lambda}^\gamma_\mathcal{T}(\boldsymbol{x}), |\boldsymbol{\lambda}^\gamma_\mathcal{T}(\boldsymbol{x})| \le j_c(\boldsymbol{x})$, $|\mathbf{E}^\gamma_\mathcal{T}(\boldsymbol{x})|\le\gamma^{-1}$, and $|\boldsymbol{\lambda}(\boldsymbol{x})| = j_c(\boldsymbol{x})$. Hence, we can derive

$$\begin{aligned}
&(\boldsymbol{\lambda}(\boldsymbol{x}) - \boldsymbol{\lambda}^\gamma_\mathcal{T}(\boldsymbol{x}))\cdot(\mathbf{E}^\gamma_\mathcal{T}(\boldsymbol{x}) - \mathbf{E}(\boldsymbol{x}))\\
&\quad= \boldsymbol{\lambda}(\boldsymbol{x})\cdot\mathbf{E}^\gamma_\mathcal{T}(\boldsymbol{x}) - j_c(\boldsymbol{x})|\mathbf{E}(\boldsymbol{x})| - \gamma j_c(\boldsymbol{x})|\mathbf{E}^\gamma_\mathcal{T}(\boldsymbol{x})|^2 + \boldsymbol{\lambda}^\gamma_\mathcal{T}(\boldsymbol{x})\cdot\mathbf{E}(\boldsymbol{x})\\
&\quad\le \frac{1}{\gamma}j_c(\boldsymbol{x}) - j_c(\boldsymbol{x})|\mathbf{E}(\boldsymbol{x})| + j_c(\boldsymbol{x})|\mathbf{E}(\boldsymbol{x})| - \gamma j_c(\boldsymbol{x})|\mathbf{E}^\gamma_\mathcal{T}|^2 \le \frac{1}{\gamma}j_c(\boldsymbol{x}).
\end{aligned}$$

For $\boldsymbol{x}\in\mathcal{I}\cap\mathcal{A}_\gamma$, we have $\mathbf{E}(\boldsymbol{x}) = 0$ and thus

$$(\boldsymbol{\lambda}(\boldsymbol{x}) - \boldsymbol{\lambda}^\gamma_\mathcal{T}(\boldsymbol{x}))\cdot(\mathbf{E}^\gamma_\mathcal{T}(\boldsymbol{x}) - \mathbf{E}(\boldsymbol{x})) = (\boldsymbol{\lambda}(\boldsymbol{x}) - \boldsymbol{\lambda}^\gamma_\mathcal{T}(\boldsymbol{x}))\cdot\mathbf{E}^\gamma_\mathcal{T}(\boldsymbol{x}) \le 0.$$

Finally, for $\boldsymbol{x} \in \mathcal{I} \cap \mathcal{I}_\gamma$, we have $\mathbf{E}(\boldsymbol{x}) = 0, j_c(\boldsymbol{x})\mathbf{E}_{\mathcal{T}}^\gamma(\boldsymbol{x}) = \gamma^{-1}\boldsymbol{\lambda}_{\mathcal{T}}^\gamma(\boldsymbol{x})$ as well as $|\mathbf{E}_{\mathcal{T}}^\gamma(\boldsymbol{x})| \leq \gamma^{-1}$. This implies that

$$(\boldsymbol{\lambda}(\boldsymbol{x}) - \boldsymbol{\lambda}_{\mathcal{T}}^\gamma(\boldsymbol{x})) \cdot (\mathbf{E}_{\mathcal{T}}^\gamma(\boldsymbol{x}) - \mathbf{E}(\boldsymbol{x})) \leq \frac{1}{\gamma} j_c(\boldsymbol{x}) - \gamma j_c(\boldsymbol{x}) |\mathbf{E}_{\mathcal{T}}^\gamma|^2 \leq \frac{1}{\gamma} j_c(\boldsymbol{x}).$$

After taking all the pointwise estimates above together, (3.8) follows by integration over Ω. Now, inserting $\mathbf{v} = \mathbf{E}_{\mathcal{T}} - \mathbf{E}$ into (3.7), we get from (3.8) that

$$\|\mathbf{E}_{\mathcal{T}}^\gamma - \mathbf{E}\|_a^2 \leq \|\mathbf{E}_{\mathcal{T}}^\gamma - \mathbf{z}\|_a \|\mathbf{E}_{\mathcal{T}}^\gamma - \mathbf{E}\|_a + \frac{1}{\gamma}\|j_c\|_{L^1(\Omega)}\,.$$

Further, the application of Young's inequality yields

$$\|\mathbf{E}_{\mathcal{T}}^\gamma - \mathbf{E}\|_a^2 \leq \|\mathbf{E}_{\mathcal{T}}^\gamma - \mathbf{z}\|_a^2 + \frac{2}{\gamma}\|j_c\|_{L^1(\Omega)}. \tag{3.9}$$

Finally, (3.6), (3.9), and the triangle inequality give us

$$\begin{aligned} \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_{\mathcal{T}}^\gamma\|_{*,a}^2 &\leq \|\mathbf{z} - \mathbf{E}\|_a^2 \leq 2\|\mathbf{z} - \mathbf{E}_{\mathcal{T}}^\gamma\|_a^2 + 2\|\mathbf{E}_{\mathcal{T}}^\gamma - \mathbf{E}\|_a^2 \\ &\leq 4\|\mathbf{z} - \mathbf{E}_{\mathcal{T}}^\gamma\|_a^2 + \frac{4}{\gamma}\|j_c\|_{L^1(\Omega)}. \end{aligned} \tag{3.10}$$

The desired assertion follows directly from (3.9) and (3.10). ∎

The next lemma gives a local regularity property of the regularized dual variable.

LEMMA 3.3. *Under Assumption* 2.1*, the dual variable from* (3.3) *enjoys the regularity property* $\boldsymbol{\lambda}_{\mathcal{T}}^\gamma|_T \in \mathbf{H}(\mathrm{div}, T)$ *for every* $T \in \mathcal{T}$ *and the following stability estimate:*

$$\|\operatorname{div} \boldsymbol{\lambda}_{\mathcal{T}}^\gamma|_T\|_{0,T} \leq \frac{\gamma\|j_c\|_{L^\infty(\Omega)}}{\sqrt{2}}\|\mathbf{curl}\,\mathbf{E}_{\mathcal{T}}^\gamma\|_{0,T} \quad \forall T \in \mathcal{T}.$$

Proof. Let $T \in \mathcal{T}$ be an arbitrarily fixed element. As $\mathbf{E}_{\mathcal{T}}^\gamma \in \mathbf{V}_{\mathcal{T}}$, it holds that $\mathbf{E}_{\mathcal{T}}^\gamma|_T \in \boldsymbol{\mathcal{C}}^\infty(T)$, and thus $\max\{1, \gamma|\mathbf{E}_{\mathcal{T}}^\gamma|_T|\} \in W^{1,\infty}(T)$ (see [26, Corollary A.6]). For this reason and the fact that $j_c|_T$ is constant, $\boldsymbol{\lambda}_{\mathcal{T}}^\gamma|_T \in \boldsymbol{W}^{1,\infty}(T)$ follows from (3.3). Now we show that

$$\operatorname{div} \boldsymbol{\lambda}_{\mathcal{T}}^\gamma|_T = \begin{cases} -\dfrac{j_c|_T}{|\mathbf{E}_{\mathcal{T}}^\gamma|_T|^3}(\nabla \mathbf{E}_{\mathcal{T}}^\gamma|_T \mathbf{E}_{\mathcal{T}}^\gamma|_T) \cdot \mathbf{E}_{\mathcal{T}}^\gamma|_T & \text{in } \mathcal{A}_\gamma \cap T, \\ 0 & \text{in } \mathcal{I}_\gamma \cap T, \end{cases} \tag{3.11}$$

where $\nabla \mathbf{E}_{\mathcal{T}}^\gamma|_T = (\nabla \mathbf{E}_{\mathcal{T}}^\gamma|_{T,1}\ \nabla \mathbf{E}_{\mathcal{T}}^\gamma|_{T,2}\ \nabla \mathbf{E}_{\mathcal{T}}^\gamma|_{T,3})$. Indeed, since j_c is piecewise constant (see (A1)) and according to (3.3) and (3.4), we may compute

$$\begin{aligned} \partial_i \boldsymbol{\lambda}_{\mathcal{T}}^\gamma|_{T,i} &= j_c|_T \partial_i \left(\frac{\gamma \mathbf{E}_{\mathcal{T}}^\gamma|_{T,i}}{\max\{1, \gamma|\mathbf{E}_{\mathcal{T}}^\gamma|_T|\}} \right) \\ &= j_c|_T \left(\frac{\gamma \partial_i \mathbf{E}_{\mathcal{T}}^\gamma|_{T,i}}{\max\{1, \gamma|\mathbf{E}_{\mathcal{T}}^\gamma|_T|\}} - \frac{\gamma \mathbf{E}_{\mathcal{T}}^\gamma|_{T,i}}{\max\{1, \gamma|\mathbf{E}_{\mathcal{T}}^\gamma|_T|\}^2} \partial_i\big(\max\{1, \gamma|\mathbf{E}_{\mathcal{T}}^\gamma|_T|\}\big) \right). \end{aligned} \tag{3.12}$$

Additionally, thanks to [26, Corollary A.6], it holds that

$$\partial_i\big(\max\{1, \gamma|\mathbf{E}_{\mathcal{T}}^\gamma|_T|\}\big) = \begin{cases} \dfrac{\gamma \mathbf{E}_{\mathcal{T}}^\gamma|_T \cdot \partial_i \mathbf{E}_{\mathcal{T}}^\gamma|_T}{|\mathbf{E}_{\mathcal{T}}^\gamma|_T|} & \text{in } \mathcal{A}_\gamma \cap T, \\ 0 & \text{in } \mathcal{I}_\gamma \cap T. \end{cases} \tag{3.13}$$

Thus, in view of (3.12), (3.13), the regularity $\boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}|_T \in \boldsymbol{W}^{1,\infty}(T)$, and $\operatorname{div} \mathbf{E}_{\mathcal{T}}^{\gamma}|_T \equiv 0$, it follows that

$$\operatorname{div} \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}|_T = \sum_{i=1}^{3} \partial_i \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}|_{T,i} = \begin{cases} -\dfrac{j_c|_T}{|\mathbf{E}_{\mathcal{T}}^{\gamma}|_T|^3} \displaystyle\sum_{i=1}^{3} \mathbf{E}_{\mathcal{T}}^{\gamma}|_{T,i} \big(\partial_i \mathbf{E}_{\mathcal{T}}^{\gamma}|_T \cdot \mathbf{E}_{\mathcal{T}}^{\gamma}|_T\big) & \text{in } \mathcal{A}_\gamma \cap T, \\ 0 & \text{in } \mathcal{I}_\gamma \cap T, \end{cases}$$

which yields that (3.11) is valid. Note that according to (3.4), (3.11) implies that $\operatorname{div} \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}|_T = 0$ a.e. in $\{\boldsymbol{x} \in T : \gamma |\mathbf{E}_{\mathcal{T}}^{\gamma}|_T(\boldsymbol{x})| = 1\}$. Next, (3.11) together with the inequality $|\mathbf{E}_{\mathcal{T}}^{\gamma}(\boldsymbol{x})| > \frac{1}{\gamma}$ for a.e. $\boldsymbol{x} \in \mathcal{A}_\gamma$ (see (3.4)) leads to

$$\begin{aligned} (3.14) \quad \|\operatorname{div} \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}|_T\|_{0,T}^2 = \|\operatorname{div} \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}|_T\|_{L^2(T\cap\mathcal{A}_\gamma)}^2 &\le \|j_c\|_{L^\infty(\Omega)}^2 \int_{T\cap\mathcal{A}_\gamma} \frac{|\nabla \mathbf{E}_{\mathcal{T}}^{\gamma}|_T(\boldsymbol{x})|^2}{|\mathbf{E}_{\mathcal{T}}^{\gamma}|_T(\boldsymbol{x})|^2}\, d\boldsymbol{x} \\ &\le \gamma^2 \|j_c\|_{L^\infty(\Omega)}^2 \|\nabla \mathbf{E}_{\mathcal{T}}^{\gamma}|_T\|_{0,T}^2. \end{aligned}$$

But, for every $\boldsymbol{x} \in T$, we know that $\mathbf{E}_{\mathcal{T}}^{\gamma}|_T(\boldsymbol{x}) = \mathbf{a}_T \times \boldsymbol{x} + \mathbf{b}_T$ for some $\mathbf{a}_T, \mathbf{b}_T \in \mathbb{R}^3$. Thus,

$$\nabla \mathbf{E}_{\mathcal{T}}^{\gamma}|_T(\boldsymbol{x}) = \begin{bmatrix} 0 & -\mathbf{a}_{T,3} & \mathbf{a}_{T,2} \\ \mathbf{a}_{T,3} & 0 & -\mathbf{a}_{T,1} \\ -\mathbf{a}_{T,2} & \mathbf{a}_{T,1} & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{curl}\, \mathbf{E}_{\mathcal{T}}^{\gamma}|_T(\boldsymbol{x}) = 2 \begin{bmatrix} \mathbf{a}_{T,1} \\ \mathbf{a}_{T,2} \\ \mathbf{a}_{T,3} \end{bmatrix},$$

which implies that $\sqrt{2}|\nabla \mathbf{E}_{\mathcal{T}}^{\gamma}|_T(\boldsymbol{x})| = |\mathbf{curl}\, \mathbf{E}_{\mathcal{T}}^{\gamma}|_T(\boldsymbol{x})|$ for all $\boldsymbol{x} \in T$. Combining this with (3.14) yields the desired estimate. ◻

Remark 3.4. Lemma 3.3 does not hold true in general for the unregularized dual variable corresponding to ($\mathrm{VI}_{\mathcal{T}}$) due to the lack of information in the inactive set.

Next, we start to investigate the a posterior error estimate of the edge element solution $(\mathbf{E}_{\mathcal{T}}^{\gamma}, \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}) \in \mathbf{V}_{\mathcal{T}} \times \mathbf{L}^\infty(\Omega)$ to the discrete system (3.3). To do so, we define for $(\mathbf{E}_{\mathcal{T}}^{\gamma}, \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma})$ the element residual $\mathbf{R}_T$ for every $T \in \mathcal{T}$ and the normal and tangential jumps across every face $F \in \mathcal{F}_{\mathcal{T}}$:

$$(3.15) \qquad \mathbf{R}_T := \mathbf{f}|_T - \epsilon \mathbf{E}_{\mathcal{T}}^{\gamma}|_T - \mathbf{curl}\, \mu^{-1}\, \mathbf{curl}\, \mathbf{E}_{\mathcal{T}}^{\gamma}|_T - \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}|_T,$$

$$(3.16) \qquad \mathbf{J}_{F,1} := [\mu^{-1}\, \mathbf{curl}\, \mathbf{E}_{\mathcal{T}}^{\gamma} \times \mathbf{n}_F] \quad \text{and} \quad J_{F,2} := [(\boldsymbol{\lambda}_{\mathcal{T}}^{\gamma} + \epsilon \mathbf{E}_{\mathcal{T}}^{\gamma}) \cdot \mathbf{n}_F].$$

For any subset $\mathcal{M}$ of elements from $\mathcal{T}$, we define its error estimator

$$(3.17) \qquad \eta_{\mathcal{T}}^2(\mathbf{E}_{\mathcal{T}}^{\gamma}, \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}, \mathbf{f}, \mathcal{M}) := \sum_{T\in\mathcal{M}} \eta_{\mathcal{T},1}^2(\mathbf{E}_{\mathcal{T}}^{\gamma}, \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}, \mathbf{f}, T) + \eta_{\mathcal{T},2}^2(\mathbf{E}_{\mathcal{T}}^{\gamma}, \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}, T),$$

where $\eta_{\mathcal{T},1}(\mathbf{E}_{\mathcal{T}}^{\gamma}, \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}, \mathbf{f}, T)$ and $\eta_{\mathcal{T},2}(\mathbf{E}_{\mathcal{T}}^{\gamma}, \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}, T)$ are given by

$$\begin{aligned} \eta_{\mathcal{T},1}^2(\mathbf{E}_{\mathcal{T}}^{\gamma}, \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}, \mathbf{f}, T) &:= h_T^2 \|\mathbf{R}_T\|_{0,T}^2 + \sum_{F\in\partial T\cap\Omega} h_F \|\mathbf{J}_{F,1}\|_{0,F}^2, \\ \eta_{\mathcal{T},2}^2(\mathbf{E}_{\mathcal{T}}^{\gamma}, \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}, T) &:= h_T^2 \|\operatorname{div} \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}\|_{0,T}^2 + \sum_{F\in\partial T\cap\Omega} h_F \|J_{F,2}\|_{0,F}^2. \end{aligned}$$

We further define an oscillation term associated with the subset $\mathcal{M}$, namely,

$$\mathrm{osc}_{\mathcal{T}}{}^2(\mathbf{E}_{\mathcal{T}}^{\gamma}, \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}, \mathbf{f}, \mathcal{M}) \coloneqq \sum_{T\in\mathcal{M}} \mathrm{osc}_{\mathcal{T}}{}^2(\mathbf{E}_{\mathcal{T}}^{\gamma}, \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}, \mathbf{f}, T)$$

with

$$\begin{aligned}\mathrm{osc}_{\mathcal{T}}{}^2(\mathbf{E}_{\mathcal{T}}^{\gamma}, \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}, \mathbf{f}, T) \coloneqq{}& h_T^2\|\mathbf{R}_T - \bar{\mathbf{R}}_T\|_{0,T}^2 + h_T^2\|\overline{\operatorname{div}\boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}}|_T - \operatorname{div}\boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}\|_{0,T}^2\\ &+ \sum_{F\in\partial T\cap\Omega} h_F\|\mathbf{J}_{F,1} - \bar{\mathbf{J}}_{F,1}\|_{0,F}^2 + h_F\|J_{F,2} - \bar{J}_{F,2}\|_{0,F}^2,\end{aligned}$$

where $\bar{\mathbf{R}}_T$, $\overline{\operatorname{div}\boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}}|_T$, $\bar{\mathbf{J}}_{F,1}$, and $\bar{J}_{F,2}$ denote the averages of $\mathbf{R}_T$, $\operatorname{div}\boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}$, $\mathbf{J}_{F,1}$, and $J_{F,2}$ over $T\in\mathcal{T}$ and $\mathcal{F}\in\mathcal{F}_{\mathcal{T}}$, respectively, i.e.,

$$\bar{\mathbf{R}}_T \coloneqq \frac{1}{|T|}\int_T \mathbf{R}_T\,d\boldsymbol{x}, \quad \overline{\operatorname{div}\boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}}|_T \coloneqq \frac{1}{|T|}\int_T \operatorname{div}\boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}\,d\boldsymbol{x}, \quad \bar{\mathbf{J}}_{F,1} \coloneqq \frac{1}{|F|}\int_F \mathbf{J}_{F,1}\,dS,$$

and analogously for $\bar{J}_{F,2}$. In the case of $\mathcal{M}=\mathcal{T}$, we simply write $\eta_{\mathcal{T}}(\mathbf{E}_{\mathcal{T}}^{\gamma}, \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}, \mathbf{f}, \mathcal{T})$ as $\eta_{\mathcal{T}}(\mathbf{E}_{\mathcal{T}}^{\gamma}, \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}, \mathbf{f})$ and $\mathrm{osc}_{\mathcal{T}}(\mathbf{E}_{\mathcal{T}}^{\gamma}, \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}, \mathbf{f}, \mathcal{T})$ as $\mathrm{osc}_{\mathcal{T}}(\mathbf{E}_{\mathcal{T}}^{\gamma}, \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}, \mathbf{f})$. Let us now recall a quasi-interpolation operator to relate $\mathbf{H}_0(\mathbf{curl})$ to the finite element space $\mathbf{V}_{\mathcal{T}}$ (see [35, Theorem 1]).

LEMMA 3.5 (Schöberl interpolation operator). *Under Assumption* 2.1*, there exists a quasi-interpolation operator* $\boldsymbol{\Pi}_{\mathcal{T}}^s\colon \mathbf{H}_0(\mathbf{curl}) \to \mathbf{V}_{\mathcal{T}}$ *such that for every* $\mathbf{v}\in\mathbf{H}_0(\mathbf{curl})$ *there exist* $\boldsymbol{\phi}\in\mathbf{H}_0^1(\Omega)$ *and* $\varphi\in H_0^1(\Omega)$ *satisfying*

$$\mathbf{v} - \boldsymbol{\Pi}_{\mathcal{T}}^s\mathbf{v} = \boldsymbol{\phi} + \nabla\varphi$$

with the stability estimates

$$\begin{aligned} h_T^{-1}\|\boldsymbol{\phi}\|_{0,T} + \|\nabla\boldsymbol{\phi}\|_{0,T} &\le C\|\mathbf{curl}\,\mathbf{v}\|_{0,\tilde{D}_T},\\ h_T^{-1}\|\varphi\|_{0,T} + \|\nabla\varphi\|_{0,T} &\le C\|\mathbf{v}\|_{0,\tilde{D}_T},\end{aligned}$$

where the constant $C>0$ *depends only on the shape of the elements in the enlarged element patch* $\tilde{D}_T \coloneqq \{T'\in\mathcal{T} \mid T'\cap D_T \neq \emptyset\}$.

We have now collected all the necessary preparations for the a posteriori error analysis. Let us begin by proving the reliability of (3.17) in the following theorem.

THEOREM 3.6. *Under Assumption* 2.1*, there exists a constant* $C>0$ *depending only on* Ω*, the shape-regularity of* $\mathcal{T}$*, and the material parameters* ϵ,μ *as well as* j_c *such that the solutions* $(\mathbf{E},\boldsymbol{\lambda})$ *and* $(\mathbf{E}_{\mathcal{T}},\boldsymbol{\lambda}_{\mathcal{T}})$ *to* (3.1) *and* (3.3) *satisfy*

$$\|\mathbf{E}-\mathbf{E}_{\mathcal{T}}^{\gamma}\|_a^2 + \|\boldsymbol{\lambda}-\boldsymbol{\lambda}_{\mathcal{T}}\|_{*,a}^2 \le C\left(\eta_{\mathcal{T}}^2(\mathbf{E}_{\mathcal{T}}^{\gamma}, \boldsymbol{\lambda}_{\mathcal{T}}^{\gamma}, \mathbf{f}) + \frac{1}{\gamma}\right).$$

Proof. We define $\mathbf{v}\coloneqq \mathbf{z}-\mathbf{E}_{\mathcal{T}}^{\gamma}\in\mathbf{H}_0(\mathbf{curl})$ and use Lemma 3.5 to decompose $\mathbf{v}-\boldsymbol{\Pi}_{\mathcal{T}}^s\mathbf{v} = \boldsymbol{\phi}+\nabla\varphi$ with $\boldsymbol{\phi}\in\mathbf{H}_0^1(\Omega)$ and $\varphi\in H_0^1(\Omega)$. Then we take (A2), (3.3), and (3.5) into account and use integration by parts, the stability estimates in Lemma 3.5, as well as the trace theorem (cf. [38, pp. 87]) to obtain

$$
\begin{aligned}
\|\mathbf{v}\|_a^2 &= a(\mathbf{v},\mathbf{v}) = a(\mathbf{z},\mathbf{v}) - a(\mathbf{E}_{\mathcal{T}}^\gamma,\mathbf{v}) = (\mathbf{f}-\boldsymbol{\lambda}_{\mathcal{T}}^\gamma,\mathbf{v})_{0,\Omega} - a(\mathbf{E}_{\mathcal{T}}^\gamma,\mathbf{v})\\
&= (\mathbf{f}-\boldsymbol{\lambda}_{\mathcal{T}}^\gamma,\mathbf{v}-\boldsymbol{\Pi}_{\mathcal{T}}^s\mathbf{v})_{0,\Omega} + (\mathbf{f}-\boldsymbol{\lambda}_{\mathcal{T}}^\gamma,\boldsymbol{\Pi}_{\mathcal{T}}^s\mathbf{v})_{0,\Omega} - a(\mathbf{E}_{\mathcal{T}}^\gamma,\mathbf{v})\\
&= (\mathbf{f}-\boldsymbol{\lambda}_{\mathcal{T}}^\gamma,\mathbf{v}-\boldsymbol{\Pi}_{\mathcal{T}}^s\mathbf{v})_{0,\Omega} - a(\mathbf{E}_{\mathcal{T}}^\gamma,\mathbf{v}-\boldsymbol{\Pi}_{\mathcal{T}}^s\mathbf{v})\\
&= (\mathbf{f}-\boldsymbol{\lambda}_{\mathcal{T}}^\gamma,\boldsymbol{\phi}+\nabla\varphi)_{0,\Omega} - a(\mathbf{E}_{\mathcal{T}}^\gamma,\boldsymbol{\phi}+\nabla\varphi)\\
&= (\mathbf{f}-\boldsymbol{\lambda}_{\mathcal{T}}^\gamma,\boldsymbol{\phi})_{0,\Omega} - a(\mathbf{E}_{\mathcal{T}}^\gamma,\boldsymbol{\phi}) - (\boldsymbol{\lambda}_{\mathcal{T}}^\gamma+\epsilon\mathbf{E}_{\mathcal{T}}^\gamma,\nabla\varphi)_{0,\Omega}\\
&= \sum_{T\in\mathcal{T}}(\mathbf{R}_T,\boldsymbol{\phi})_{0,T} - \sum_{F\in\mathcal{F}_{\mathcal{T}}}(\mathbf{J}_{F,1},\boldsymbol{\phi})_{0,F} - (\boldsymbol{\lambda}_{\mathcal{T}}^\gamma+\epsilon\mathbf{E}_{\mathcal{T}}^\gamma,\nabla\varphi)_{0,\Omega}\\
&\le \sum_{T\in\mathcal{T}} h_T\|\mathbf{R}_T\|_{0,T}h_T^{-1}\|\boldsymbol{\phi}\|_{0,T} + \sum_{F\in\mathcal{F}_{\mathcal{T}}} h_F^{1/2}\|\mathbf{J}_{F,1}\|_{0,F}h_F^{-1/2}\|\boldsymbol{\phi}\|_{0,F}\\
&\quad - (\boldsymbol{\lambda}_{\mathcal{T}}^\gamma+\epsilon\mathbf{E}_{\mathcal{T}}^\gamma,\nabla\varphi)_{0,\Omega}\\
&\le C\sum_{T\in\mathcal{T}}\eta_{\mathcal{T},1}(\mathbf{E}_{\mathcal{T}}^\gamma,\boldsymbol{\lambda}_{\mathcal{T}}^\gamma,\mathbf{f},T)(h_T^{-1}\|\boldsymbol{\phi}\|_{0,T}+\|\nabla\boldsymbol{\phi}\|_{0,T})\\
&\quad - (\boldsymbol{\lambda}_{\mathcal{T}}^\gamma+\epsilon\mathbf{E}_{\mathcal{T}}^\gamma,\nabla\varphi)_{0,\Omega}\\
&\le C\sum_{T\in\mathcal{T}}\eta_{\mathcal{T},1}(\mathbf{E}_{\mathcal{T}}^\gamma,\boldsymbol{\lambda}_{\mathcal{T}}^\gamma,\mathbf{f},T)\|\operatorname{\mathbf{curl}}\mathbf{v}\|_{0,\tilde{D}_T} - (\boldsymbol{\lambda}_{\mathcal{T}}^\gamma+\epsilon\mathbf{E}_{\mathcal{T}}^\gamma,\nabla\varphi)_{0,\Omega}\,.
\end{aligned}
\tag{3.18}
$$

For the last term on the right-hand side of (3.18), we apply integration by parts, using Lemma 3.3 and the fact that $\operatorname{div}\epsilon\mathbf{E}_{\mathcal{T}}^\gamma|_T = 0$ over every $T\in\mathcal{T}$, as well as the trace theorem and the stability estimates from Lemma 3.5 to obtain

$$
\begin{aligned}
&-(\boldsymbol{\lambda}_{\mathcal{T}}^\gamma+\epsilon\mathbf{E}_{\mathcal{T}}^\gamma,\nabla\varphi)_{0,\Omega}\\
&\quad= \sum_{T\in\mathcal{T}}(\operatorname{div}\boldsymbol{\lambda}_{\mathcal{T}}^\gamma,\varphi)_{0,T} - \sum_{F\in\mathcal{F}_{\mathcal{T}}}(J_{F,2},\varphi)_{0,F}\\
&\quad\le C\sum_{T\in\mathcal{T}} h_T\|\operatorname{div}\boldsymbol{\lambda}_{\mathcal{T}}^\gamma\|_{0,T}h_T^{-1}\|\varphi\|_{0,T} + \sum_{F\in\mathcal{F}_{\mathcal{T}}} h_F^{1/2}\|J_{F,2}\|_{0,F}h_F^{-1/2}\|\varphi\|_{0,F}\\
&\quad\le C\sum_{T\in\mathcal{T}}\eta_{\mathcal{T},2}(\mathbf{E}_{\mathcal{T}},\boldsymbol{\lambda}_{\mathcal{T}},T)(h_T^{-1}\|\varphi\|_{0,T}+\|\nabla\varphi\|_{0,T})\\
&\quad\le C\sum_{T\in\mathcal{T}}\eta_{\mathcal{T},2}(\mathbf{E}_{\mathcal{T}},\boldsymbol{\lambda}_{\mathcal{T}},T)\|\mathbf{v}\|_{0,\tilde{D}_T}.
\end{aligned}
\tag{3.19}
$$

Finally, by inserting (3.19) into (3.18) and making use of the finite overlapping property of elements in $\tilde{D}_T$ as well as the equivalence between $\|\cdot\|_a$ and $\|\cdot\|_{\mathbf{H}(\mathbf{curl})}$, we deduce that

$$
\begin{aligned}
\|\mathbf{z}-\mathbf{E}_{\mathcal{T}}^\gamma\|_a^2 &\le C\sum_{T\in\mathcal{T}_k}\eta_{\mathcal{T}}(\mathbf{E}_{\mathcal{T}}^\gamma,\boldsymbol{\lambda}_{\mathcal{T}}^\gamma,\mathbf{f},T)\|\mathbf{z}-\mathbf{E}_{\mathcal{T}}^\gamma\|_{a,\tilde{D}_T} \le C\eta_{\mathcal{T}}(\mathbf{E}_{\mathcal{T}}^\gamma,\boldsymbol{\lambda}_{\mathcal{T}}^\gamma,\mathbf{f})\|\mathbf{z}-\mathbf{E}_{\mathcal{T}}^\gamma\|_a\\
&\overset{\text{Lemma 3.2}}{\Longrightarrow} \|\mathbf{E}-\mathbf{E}_{\mathcal{T}}^\gamma\|_a^2 + \|\boldsymbol{\lambda}-\boldsymbol{\lambda}_{\mathcal{T}}\|_{*,a}^2 \le C\left(\eta_{\mathcal{T}}^2(\mathbf{E}_{\mathcal{T}}^\gamma,\boldsymbol{\lambda}_{\mathcal{T}}^\gamma,\mathbf{f}) + \frac{1}{\gamma}\right),
\end{aligned}
$$

where the constant $C>0$ depends only on Ω,ϵ,μ,j_c and the shape-regularity of $\mathcal{T}$. ∎

The next theorem establishes the efficiency of the a posteriori error estimator.

THEOREM 3.7. *Under Assumption* 2.1*, there exists a constant* $C > 0$ *depending only on the shape-regularity of* $\mathcal{T}$ *and the material parameters such that the solutions* $(\mathbf{E}, \boldsymbol{\lambda})$ *and* $(\mathbf{E}_\mathcal{T}, \boldsymbol{\lambda}_\mathcal{T})$ *to* (3.1) *and* (3.3) *fulfill*

$$C\eta_\mathcal{T}^2(\mathbf{E}_\mathcal{T}^\gamma, \boldsymbol{\lambda}_\mathcal{T}^\gamma, \mathbf{f}, T) \le \|\mathbf{E} - \mathbf{E}_\mathcal{T}^\gamma\|_{a,\omega_T}^2 + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_\mathcal{T}^\gamma\|_{*,a,\omega_T}^2 + \operatorname{osc}_\mathcal{T}{}^2(\mathbf{E}_\mathcal{T}^\gamma, \boldsymbol{\lambda}_\mathcal{T}^\gamma, \mathbf{f}, \omega_T) \quad \forall T \in \mathcal{T}. \tag{3.20}$$

Proof. To prove this result, we use the well-known tetrahedral bubble functions b_T for every given $T \in \mathcal{T}$ and their associated estimates (see [1, page 23]). We choose $\mathbf{v} = \mathbf{v}_T = \bar{\mathbf{R}}_T b_T \in \mathbf{H}_0^1(T)$ and obtain by means of (3.1) and integration by parts

$$\begin{aligned}
C\|\bar{\mathbf{R}}_T\|_{0,T}^2 &\le \big(\bar{\mathbf{R}}_T, \mathbf{v}_T\big)_{0,T} = \big(\bar{\mathbf{R}}_T - \mathbf{R}_T, \mathbf{v}_T\big)_{0,T} + (\mathbf{R}_T, \mathbf{v}_T)_{0,T} \\
&= \big(\mathbf{f} - \mathbf{curl}\,\mu^{-1}\,\mathbf{curl}\,\mathbf{E}_\mathcal{T}^\gamma - \epsilon\mathbf{E}_\mathcal{T}^\gamma - \boldsymbol{\lambda}_\mathcal{T}^\gamma, \mathbf{v}_T\big)_{0,T} + \big(\bar{\mathbf{R}}_T - \mathbf{R}_T, \mathbf{v}_T\big)_{0,T} \\
&= (\epsilon(\mathbf{E} - \mathbf{E}_\mathcal{T}^\gamma), \mathbf{v}_T)_{0,T} + (\boldsymbol{\lambda} - \boldsymbol{\lambda}_\mathcal{T}^\gamma, \mathbf{v}_T)_{0,T} \\
&\quad + \big(\mu^{-1}\,\mathbf{curl}(\mathbf{E} - \mathbf{E}_\mathcal{T}^\gamma), \mathbf{curl}\,\mathbf{v}_T\big)_{0,T} + \big(\bar{\mathbf{R}}_T - \mathbf{R}_T, \mathbf{v}_T\big)_{0,T} \\
&\le C\|\mathbf{E} - \mathbf{E}_\mathcal{T}^\gamma\|_{a,T}\|\mathbf{v}_T\|_{\mathbf{H}(\mathbf{curl},T)} + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_\mathcal{T}^\gamma\|_{*,a,T}\|\mathbf{v}_T\|_{\mathbf{H}(\mathbf{curl},T)} \\
&\quad + \|\bar{\mathbf{R}}_T - \mathbf{R}_T\|_{0,T}\|\mathbf{v}_T\|_{0,T}.
\end{aligned}$$

Now the estimates for $\mathbf{v}_T$ [1, Theorem 2.2] give us

$$\|\mathbf{v}_T\|_{0,T} + h_T\|\,\mathbf{curl}\,\mathbf{v}_T\|_{0,T} \le C\|\bar{\mathbf{R}}_T\|_{0,T}, \tag{3.21}$$

which, together with the triangle inequality, implies that

$$Ch_T^2\|\mathbf{R}_T\|_{0,T}^2 \le \|\mathbf{E} - \mathbf{E}_\mathcal{T}^\gamma\|_{a,T}^2 + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_\mathcal{T}^\gamma\|_{*,a,T}^2 + h_T^2\|\bar{\mathbf{R}}_T - \mathbf{R}_T\|_{0,T}^2. \tag{3.22}$$

Next, we set $v = v_T = \overline{\operatorname{div}\boldsymbol{\lambda}_\mathcal{T}^\gamma}|_T\, b_T \in H_0^1(T)$ and obtain by (3.1) and $\operatorname{div}\mathbf{E}_\mathcal{T}^\gamma|_T = 0$ for every $T \in \mathcal{T}$ that

$$\begin{aligned}
C\|\overline{\operatorname{div}\boldsymbol{\lambda}_\mathcal{T}^\gamma}|_T\|_{0,T}^2 &\le \Big(\overline{\operatorname{div}\boldsymbol{\lambda}_\mathcal{T}^\gamma}|_T, v_T\Big)_{0,T} \\
&= (\operatorname{div}\boldsymbol{\lambda}_\mathcal{T}^\gamma, v_T)_{0,T} + \Big(\overline{\operatorname{div}\boldsymbol{\lambda}_\mathcal{T}^\gamma}|_T - \operatorname{div}\boldsymbol{\lambda}_\mathcal{T}^\gamma, v_T\Big)_{0,T} \\
&= (\boldsymbol{\lambda}_\mathcal{T}^\gamma, \nabla v_T)_{0,T} + \Big(\overline{\operatorname{div}\boldsymbol{\lambda}_\mathcal{T}^\gamma}|_T - \operatorname{div}\boldsymbol{\lambda}_\mathcal{T}^\gamma, v_T\Big)_{0,T} \\
&= (\boldsymbol{\lambda}_\mathcal{T}^\gamma - \boldsymbol{\lambda}, \nabla v_T)_{0,T} + (\epsilon(\mathbf{E}_\mathcal{T}^\gamma - \mathbf{E}), \nabla v_T)_{0,T} + \Big(\overline{\operatorname{div}\boldsymbol{\lambda}_\mathcal{T}^\gamma}|_T - \operatorname{div}\boldsymbol{\lambda}_\mathcal{T}^\gamma, v_T\Big)_{0,T}.
\end{aligned}$$

Hence, the estimates for v_T yield

$$Ch_T^2\|\operatorname{div}\boldsymbol{\lambda}_\mathcal{T}^\gamma\|_{0,T}^2 \le \|\mathbf{E} - \mathbf{E}_\mathcal{T}^\gamma\|_{a,T}^2 + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_\mathcal{T}^\gamma\|_{*,a,T}^2 + h_T^2\|\overline{\operatorname{div}\boldsymbol{\lambda}_\mathcal{T}^\gamma}|_T - \operatorname{div}\mathbf{E}_\mathcal{T}^\gamma\|_{0,T}^2. \tag{3.23}$$

For a face $F \in \mathcal{F}_\mathcal{T}$, we use the face bubble function b_F [1] and set $\mathbf{v} = \mathbf{v}_F = \bar{\mathbf{J}}_{F,1} b_F \in \mathbf{H}_0^1(\omega_F)$. Then similar arguments yield

$$\begin{aligned}
C\|\bar{\mathbf{J}}_{F,1}\|_{0,F}^2 &\le \big(\bar{\mathbf{J}}_{F,1}, \mathbf{v}_F\big)_{0,F} = (\mathbf{J}_{F,1}, \mathbf{v}_F)_{0,F} + \big(\bar{\mathbf{J}}_{F,1} - \mathbf{J}_{F,1}, \mathbf{v}_F\big)_{0,F} \\
&= (\mathbf{R}_T, \mathbf{v}_F)_{0,\omega_F} - (\mathbf{f} - \epsilon\mathbf{E}_\mathcal{T}^\gamma - \boldsymbol{\lambda}_\mathcal{T}^\gamma, \mathbf{v}_F)_{0,\omega_F} + \big(\mu^{-1}\,\mathbf{curl}\,\mathbf{E}_\mathcal{T}^\gamma, \mathbf{curl}\,\mathbf{v}_F\big)_{0,\omega_F} \\
&\quad + \big(\bar{\mathbf{J}}_{F,1} - \mathbf{J}_{F,1}, \mathbf{v}_F\big)_{0,F} \\
&= (\mathbf{R}_T, \mathbf{v}_F)_{0,\omega_F} - (\epsilon(\mathbf{E} - \mathbf{E}_\mathcal{T}^\gamma), \mathbf{v}_F)_{0,\omega_F} - (\boldsymbol{\lambda} - \boldsymbol{\lambda}_\mathcal{T}^\gamma, \mathbf{v}_F)_{0,\omega_F} \\
&\quad - \big(\mu^{-1}\,\mathbf{curl}(\mathbf{E} - \mathbf{E}_\mathcal{T}^\gamma), \mathbf{curl}\,\mathbf{v}_F\big)_{0,\omega_F} + \big(\bar{\mathbf{J}}_{F,1} - \mathbf{J}_{F,1}, \mathbf{v}_F\big)_{0,F}.
\end{aligned}$$

We use the estimates for $\mathbf{v}_F$ [1, Theorem 2.4] again, along with (3.22), to obtain

$$Ch_F\|\mathbf{J}_{F,1}\|_{0,F}^2 \le \sum_{T\in\omega_F}\Big(\|\mathbf{E}-\mathbf{E}_\mathcal{T}^\gamma\|_{\mathbf{H}(\mathbf{curl},T)}^2 + \|\boldsymbol{\lambda}-\boldsymbol{\lambda}_\mathcal{T}^\gamma\|_{*,T}^2 + h_T^2\|\bar{\mathbf{R}}_T-\mathbf{R}_T\|_{0,T}^2\Big) + h_F\|\bar{\mathbf{J}}_{F,1}-\mathbf{J}_{F,1}\|_{0,F}^2. \tag{3.24}$$

Next, we set $q = q_F = \bar{J}_{F,2}b_F \in H_0^1(\omega_F)$ to derive analogously

$$\begin{aligned}C\|\bar{J}_{F,2}\|_{0,F}^2 &\le \big(\bar{J}_{F,2}, q_F\big)_{0,F} = (J_{F,2}, q_F)_{0,F} + \big(\bar{J}_{F,2}-J_{F,2}, q_F\big)_{0,F}\\
&= (\epsilon(\mathbf{E}-\mathbf{E}_\mathcal{T}^\gamma), \nabla q_F)_{0,\omega_F} + (\boldsymbol{\lambda}-\boldsymbol{\lambda}_\mathcal{T}^\gamma, \nabla q_F)_{0,\omega_F} + (\operatorname{div}\boldsymbol{\lambda}_\mathcal{T}^\gamma, q_F)_{0,\omega_F}\\
&\quad + \big(\bar{J}_{F,2}-J_{F,2}, q_F\big)_{0,F}\\
&\le C\Bigg(h_F^{-1/2}\sum_{T\in\omega_F}\|\mathbf{E}-\mathbf{E}_\mathcal{T}^\gamma\|_{0,T} + \|\boldsymbol{\lambda}-\boldsymbol{\lambda}_\mathcal{T}^\gamma\|_{*,a,T}\\
&\qquad + h_F^{1/2}\sum_{T\in\omega_F}\|\operatorname{div}\boldsymbol{\lambda}_\mathcal{T}^\gamma\|_{0,T} + h_F^{1/2}\|\bar{J}_{F,2}-J_{F,2}\|_{0,F}\Bigg)\|\bar{J}_{F,2}\|_{0,F}.\end{aligned}$$

Hence,

$$\begin{aligned}Ch_F\|J_{F,2}\|_{0,F}^2 \le \sum_{T\in\omega_F}&\|\mathbf{E}-\mathbf{E}_\mathcal{T}^\gamma\|_{0,T}^2 + \|\boldsymbol{\lambda}-\boldsymbol{\lambda}_\mathcal{T}^\gamma\|_{*,a,T}^2 + h_T^2\|\overline{\operatorname{div}\boldsymbol{\lambda}_\mathcal{T}^\gamma}|_T - \operatorname{div}\boldsymbol{\lambda}_\mathcal{T}^\gamma\|_{0,T}^2\\
&+ h_F\|\bar{J}_{F,2}-J_{F,2}\|_{0,F}^2.\end{aligned}$$

This, along with (3.22)–(3.24), leads directly to the efficiency of the estimator (3.20). ∎

4. Adaptive algorithm and its convergence.

4. Adaptive algorithm and its convergence. This section is devoted to the development of an adaptive mesh refinement algorithm for solving elliptic VIs of the second kind and a rigorous convergence analysis thereof. The algorithm is based on the reliable and efficient a posteriori error estimator (3.17).

While we were using the subscript $\mathcal{T}$ to indicate the finite element spaces in the previous section, we will now work with triangulations generated by our new adaptive mesh refinement algorithm. So it will be more convenient for us to indicate the dependencies on the triangulations by the number of refinement steps $k \in \mathbb{N}_0$.

Algorithm 4.1 Adaptive mesh refinement algorithm.

1: Set $k = 0$, and choose an initial conforming mesh $\mathcal{T}_0$
2: (SOLVE) Compute the solution $(\mathbf{E}_k, \boldsymbol{\lambda}_k)$ of (3.3) for $\mathcal{T} = \mathcal{T}_k, \gamma = \gamma_k$
3: (ESTIMATE) Compute the error estimator $\eta_\mathcal{T}(\mathbf{E}_k, \boldsymbol{\lambda}_k, \mathbf{f})$ defined in (3.17)
4: (MARK) Mark a subset $\mathcal{M}_k \subset \mathcal{T}_k$ containing at least the element $\tilde{T} \in \mathcal{T}_k$ with the largest error indicator, i.e.,

$$\eta_k(\mathbf{E}_k, \boldsymbol{\lambda}_k, \mathbf{f}, \tilde{T}) = \max_{T\in\mathcal{T}_k}\eta_k(\mathbf{E}_k, \boldsymbol{\lambda}_k, \mathbf{f}, T). \tag{4.1}$$

5: (REFINE) Refine each $T \in \mathcal{M}_k$ by bisection to obtain $\mathcal{T}_{k+1}$
6: Set $k = k + 1$ and go to step 2 unless a stopping criterion is satisfied

Remark 4.1.

(i) The Moreau–Yosida regularization enables us to accomplish step 2 of Algorithm 4.1 by using the semismooth Newton method [25].

(ii) We emphasize that many practical marking strategies satisfy (4.1), including the maximum strategy [2], the equidistribution strategy [21], the modified equidistribution strategy, as well as Dörfler's strategy [20].

We shall choose the sequence of Moreau–Yosida regularization parameters such that

$$\lim_{k\to\infty} \gamma_k = \infty. \tag{4.2}$$

For instance, we may set $\gamma_k = \sqrt{|\mathcal{T}_k|} + \gamma_0$, where $\gamma_0 > 0$ and $|\mathcal{T}_k|$ denotes the number of elements in $\mathcal{T}_k$.

Our aim now is to prove the convergence of Algorithm 4.1. As a first result in this section, we establish a stability estimate for the a posteriori error estimator.

LEMMA 4.2. *Let Assumption* 2.1 *hold and* $\{(\mathbf{E}_k, \boldsymbol{\lambda}_k)\}_{k\in\mathbb{N}_0}$ *be the sequence generated by Algorithm* 4.1. *Then there exists a constant* $C > 0$ *independent of* $k \in \mathbb{N}_0$ *such that for every* $T \in \mathcal{T}_k$,

$$\eta_k(\mathbf{E}_k, \boldsymbol{\lambda}_k, \mathbf{f}, T) \le C\big(\|\mathbf{E}_k\|_{0,\omega_T} + (1+\gamma_k h_T)\|\,\mathbf{curl}\,\mathbf{E}_k\|_{0,\omega_T} + \|\boldsymbol{\lambda}_k\|_{0,\omega_T} + h_T\|\mathbf{f}\|_{0,T}\big).$$

Proof. Using the fact that $\mathbf{curl}\,\mu^{-1}\,\mathbf{curl}\,\mathbf{E}_k|_T \equiv 0$ holds for all $T \in \mathcal{T}_k$, we have

$$h_T\|\mathbf{R}_T\|_{0,T} \le C h_T\left(\|\mathbf{f}\|_{0,T} + \|\mathbf{E}_k\|_{0,T} + \|\boldsymbol{\lambda}_k\|_{0,T}\right). \tag{4.3}$$

Further, by the trace theorem [38, page 87], we can estimate the tangential and normal jump terms across a face $F \in \mathcal{F}_k(\Omega)$ shared by $T, T' \in \mathcal{T}_k$, respectively, by

$$\begin{aligned} h_F^{1/2}\|\mathbf{J}_{F,1}\|_{0,F} &\le C h_F^{1/2}\left(\|\,\mathbf{curl}\,\mathbf{E}_k|_T\|_{0,F} + \|\,\mathbf{curl}\,\mathbf{E}_k|_{T'}\|_{0,F}\right) \\ &\le C\|\,\mathbf{curl}\,\mathbf{E}_k\|_{0,\omega_F}, \end{aligned} \tag{4.4}$$

$$h_F^{1/2}\|J_{F,2}\|_{0,F} \le C(\|\mathbf{E}_k\|_{0,\omega_F} + \|\boldsymbol{\lambda}_k\|_{0,\omega_F}). \tag{4.5}$$

Now the desired estimate follows directly from (4.3)–(4.5) and Lemma 3.3. ◻

To proceed with the convergence analysis, we introduce the following limiting problem: Find $\mathbf{E}_\infty \in \mathbf{V}_\infty$ such that

$$a(\mathbf{E}_\infty, \mathbf{v}_\infty - \mathbf{E}_\infty) + \varphi(\mathbf{v}_\infty) - \varphi(\mathbf{E}_\infty) \ge \int_\Omega \mathbf{f}\cdot(\mathbf{v}_\infty - \mathbf{E}_\infty)\,d\boldsymbol{x} \quad \forall\, \mathbf{v}_\infty \in \mathbf{V}_\infty, \tag{VI$_\infty$}$$

where $\mathbf{V}_\infty$ is a limiting space formed by the discrete spaces $\mathbf{V}_k$ generated by Algorithm 4.1, namely,

$$\mathbf{V}_\infty := \overline{\bigcup_{k\in\mathbb{N}_0} \mathbf{V}_k}^{\|\cdot\|_{\mathbf{H}(\mathbf{curl})}}. \tag{4.6}$$

Since $\mathbf{V}_\infty$ is a closed subspace of $\mathbf{H}(\mathbf{curl})$, the existence and uniqueness of the solutions to (VI$_\infty$) follows again by [28, Theorem 2.2]. The next lemma shows the existence of a corresponding Lagrange multiplier.

LEMMA 4.3. *Under Assumption* 2.1, *there exists a corresponding Lagrange multiplier* $\boldsymbol{\lambda}_\infty \in \mathbf{L}^\infty(\Omega)$ *for the solution* $\mathbf{E}_\infty \in \mathbf{V}_\infty$ *to* (VI$_\infty$) *such that*

$$\begin{cases} a(\mathbf{E}_\infty, \mathbf{v}_\infty) + \displaystyle\int_\Omega \boldsymbol{\lambda}_\infty \cdot \mathbf{v}_\infty \, d\boldsymbol{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v}_\infty \, d\boldsymbol{x} \quad \forall \mathbf{v}_\infty \in \mathbf{V}_\infty \\ |\boldsymbol{\lambda}_\infty(\boldsymbol{x})| \le j_c(\boldsymbol{x}), \quad \boldsymbol{\lambda}_\infty(\boldsymbol{x}) \cdot \mathbf{E}_\infty(\boldsymbol{x}) = j_c(\boldsymbol{x})|\mathbf{E}_\infty(\boldsymbol{x})| \ \textit{for a.e.}\ \boldsymbol{x} \in \Omega. \end{cases} \tag{4.7}$$

Proof. For the convenience of the reader, we provide a quick proof for this result. Choosing $\mathbf{v}_\infty = 0$ and $\mathbf{v}_\infty = 2\mathbf{E}_\infty$, respectively, in (VI$_\infty$) yields

$$a(\mathbf{E}_\infty, \mathbf{E}_\infty) + \varphi(\mathbf{E}_\infty) = \int_\Omega \mathbf{f} \cdot \mathbf{E}_\infty \, d\boldsymbol{x}. \tag{4.8}$$

Applying this identity to (VI$_\infty$) leads to

$$\int_\Omega \mathbf{f} \cdot \mathbf{v}_\infty \, d\boldsymbol{x} - a(\mathbf{E}_\infty, \mathbf{v}_\infty) =: l(\mathbf{v}_\infty) \le \varphi(\mathbf{v}_\infty) \underbrace{=}_{(1.1)} \int_\Omega j_c |\mathbf{v}_\infty| \, d\boldsymbol{x} \quad \forall\, \mathbf{v}_\infty \in \mathbf{V}_\infty.$$

As $\mathbf{V}_\infty \subset \mathbf{L}^2(\Omega)$ is a subspace, $l : \mathbf{V}_\infty \to \mathbb{R}$ is a linear functional, and $\varphi : \mathbf{L}^2(\Omega) \to \mathbb{R}$ is sublinear, the Hahn–Banach theorem implies the existence of a linear extension $F : \mathbf{L}^2(\Omega) \to \mathbb{R}$ such that

$$F(\mathbf{v}_\infty) = l(\mathbf{v}_\infty) \quad \forall \mathbf{v}_\infty \in \mathbf{V}_\infty\,; \quad |F(\mathbf{v})| \le \varphi(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{L}^2(\Omega). \tag{4.9}$$

By the boundedness of $\varphi : \mathbf{L}^2(\Omega) \to \mathbb{R}$, the Riesz representation theorem yields the existence of $\boldsymbol{\lambda}_\infty \in \mathbf{L}^2(\Omega)$ satisfying

$$F(\mathbf{v}) = (\boldsymbol{\lambda}_\infty, \mathbf{v})_{\mathbf{L}^2(\Omega)} \quad \forall \mathbf{v} \in \mathbf{L}^2(\Omega).$$

Thus, the equation in (4.9) is equivalent to

$$a(\mathbf{E}_\infty, \mathbf{v}_\infty) + \int_\Omega \boldsymbol{\lambda}_\infty \cdot \mathbf{v}_\infty \, d\boldsymbol{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v}_\infty \, d\boldsymbol{x} \quad \forall \mathbf{v}_\infty \in \mathbf{V}_\infty. \tag{4.10}$$

Assume now that there exists a measurable set $\omega \subset \Omega$ with $|\omega| \neq 0$ such that $|\boldsymbol{\lambda}_\infty(\boldsymbol{x})| > j_c(\boldsymbol{x})$ for a.e. $\boldsymbol{x} \in \omega$. By this assumption, the function $\hat{\mathbf{v}} := \frac{\boldsymbol{\lambda}_\infty}{|\boldsymbol{\lambda}_\infty|}\chi_\omega$ belongs to $\mathbf{L}^2(\Omega)$. Then taking $\mathbf{v} = \hat{\mathbf{v}}$ in the inequality in (4.9) leads readily to a contradiction

$$\int_\omega j_c \, d\boldsymbol{x} < \int_\omega |\boldsymbol{\lambda}_\infty| \, d\boldsymbol{x} \le \int_\omega j_c \, d\boldsymbol{x}.$$

Thus,

$$|\boldsymbol{\lambda}_\infty(\boldsymbol{x})| \le j_c(\boldsymbol{x}) \text{ a.e. } \boldsymbol{x} \in \Omega \quad \Rightarrow \quad \boldsymbol{\lambda}_\infty \in \mathbf{L}^\infty(\Omega). \tag{4.11}$$

Finally, inserting $\mathbf{v}_\infty = \mathbf{E}_\infty$ into (4.10), we deduce from (4.8) that

$$\int_\Omega j_c(\boldsymbol{x})|\mathbf{E}_\infty(\boldsymbol{x})| - \boldsymbol{\lambda}_\infty(\boldsymbol{x}) \cdot \mathbf{E}_\infty(\boldsymbol{x}) \, d\boldsymbol{x} = 0$$

$$\underbrace{\Rightarrow}_{(4.11)} \boldsymbol{\lambda}_\infty(\boldsymbol{x}) \cdot \mathbf{E}_\infty(\boldsymbol{x}) = j_c(\boldsymbol{x})|\mathbf{E}_\infty(\boldsymbol{x})| \text{ for a.e. } \boldsymbol{x} \in \Omega.$$

In conclusion, $(\mathbf{E}_\infty, \boldsymbol{\lambda}_\infty) \in \mathbf{V}_\infty \times \mathbf{L}^\infty(\Omega)$ satisfies (4.7). ∎

Next, we prove the strong convergence of the edge element sequence $\{\mathbf{E}_k\}_{k\in\mathbb{N}_0}$ towards the unique solution of (VI_∞).

THEOREM 4.4. *Let Assumption* 2.1 *hold. Moreover, let* $\mathbf{E}_\infty \in \mathbf{V}_\infty$ *be the unique solution to* (VI_∞) *and* $\{\mathbf{E}_k\}_{k\in\mathbb{N}_0}$ *be the sequence generated by Algorithm* 4.1*. Then the following convergence holds:*

$$\lim_{k\to\infty} \|\mathbf{E}_k - \mathbf{E}_\infty\|_a = 0.$$

Proof. Let us begin this proof by showing that the sequence $\{\mathbf{E}_k\}_{k\in\mathbb{N}_0} \subset \mathbf{H}(\mathbf{curl})$ is bounded. Since $\mathbf{E}_k$ solves $(\mathrm{VI}_{\mathcal{T}})$, we obtain

$$a(\mathbf{E}_k, \mathbf{E}_k) \le (\mathbf{f}, \mathbf{E}_k - \mathbf{v}_k)_{0,\Omega} + a(\mathbf{E}_k, \mathbf{v}_k) + \varphi_{\gamma_k}(\mathbf{v}_k) - \varphi_{\gamma_k}(\mathbf{E}_k) \quad \forall \mathbf{v}_k \in \mathbf{V}_k. \tag{4.12}$$

Then, setting $\mathbf{v}_k = 0$ in (4.12), we derive by Hölder's inequality and Lemma 3.1 that

$$\|\mathbf{E}_k\|_a^2 \le C(\|\mathbf{E}_k\|_a + 1) \quad \Rightarrow \quad \|\mathbf{E}_k\|_a \le C,$$

where the constant $C > 0$ is independent of k. Hence, there exists a $\mathbf{w}_\infty \in \mathbf{V}_\infty$ and a subsequence of $\{\mathbf{E}_k\}_{k\in\mathbb{N}_0}$, still denoted by $\{\mathbf{E}_k\}_{k\in\mathbb{N}_0}$, such that

$$\mathbf{E}_k \rightharpoonup \mathbf{w}_\infty \text{ weakly in } \mathbf{H}_0(\mathbf{curl}) \text{ as } k \to \infty. \tag{4.13}$$

By exploiting the weak lower semicontinuity of the squared norm $\|\cdot\|_a^2$, we obtain

$$a(\mathbf{w}_\infty, \mathbf{w}_\infty) \le \liminf_{k\to\infty} a(\mathbf{E}_k, \mathbf{E}_k). \tag{4.14}$$

Now fix $\mathbf{v}_\infty \in \mathbf{V}_\infty$. Thanks to (4.6), we may find a sequence $\{\mathbf{v}_k\}_{k\in\mathbb{N}_0}$ such that $\mathbf{v}_k \in \mathbf{V}_k$ for every $k \in \mathbb{N}_0$ and $\mathbf{v}_k \to \mathbf{v}_\infty$ in $\mathbf{H}(\mathbf{curl})$ as $k \to \infty$. Thus, (3.2), Lemma 3.1 with (4.2), (4.13), and (4.14) lead to

$$\begin{aligned}
&(\mathbf{f}, \mathbf{v}_\infty - \mathbf{w}_\infty)_{0,\Omega} \\
&\quad = \lim_{k\to\infty} (\mathbf{f}, \mathbf{v}_k - \mathbf{E}_k)_{0,\Omega} \\
&\quad \le \limsup_{k\to\infty} \left[a(\mathbf{E}_k, \mathbf{v}_k - \mathbf{E}_k) + \varphi_{\gamma_k}(\mathbf{v}_k) - \varphi_{\gamma_k}(\mathbf{E}_k)\right] \\
&\quad \le \limsup_{k\to\infty} a(\mathbf{E}_k, \mathbf{v}_k) - \liminf_{k\to\infty} a(\mathbf{E}_k, \mathbf{E}_k) + \limsup_{k\to\infty} \varphi_{\gamma_k}(\mathbf{v}_k) - \liminf_{k\to\infty} \varphi_{\gamma_k}(\mathbf{E}_k) \\
&\quad \le a(\mathbf{w}_\infty, \mathbf{v}_\infty - \mathbf{w}_\infty) + \varphi(\mathbf{v}_\infty) - \varphi(\mathbf{w}_\infty).
\end{aligned}$$

Since $\mathbf{v}_\infty \in \mathbf{V}_\infty$ was chosen arbitrarily, the uniqueness of the solution to (VI_∞) implies that $\mathbf{w}_\infty = \mathbf{E}_\infty$ and $\mathbf{E}_k \rightharpoonup \mathbf{E}_\infty$ in $\mathbf{H}_0(\mathbf{curl})$ as $k \to \infty$.

To further show the strong convergence, we consider $\{\mathbf{v}_k\}_{k\in\mathbb{N}_0}$ such that $\mathbf{v}_k \in \mathbf{V}_k$ for every $k \in \mathbb{N}_0$ and $\mathbf{v}_k \to \mathbf{E}_\infty$ in $\mathbf{H}(\mathbf{curl})$ as $k \to \infty$. The existence of such a sequence follows by the definition of $\mathbf{V}_\infty$ in (4.6). Therefore, we deduce by means of (4.12) that

$$\begin{aligned}
0 \le \|\mathbf{E}_k - \mathbf{E}_\infty\|_a^2 &\le a(\mathbf{E}_k, \mathbf{E}_k) - a(\mathbf{E}_k, \mathbf{E}_\infty) - a(\mathbf{E}_\infty, \mathbf{E}_k - \mathbf{E}_\infty) \le (\mathbf{f}, \mathbf{E}_k - \mathbf{v}_k)_{0,\Omega} \\
&\quad + a(\mathbf{E}_k, \mathbf{v}_k) + \varphi_{\gamma_k}(\mathbf{v}_k) - \varphi_{\gamma_k}(\mathbf{E}_k) - a(\mathbf{E}_k, \mathbf{E}_\infty) - a(\mathbf{E}_\infty, \mathbf{E}_k - \mathbf{E}_\infty).
\end{aligned}$$

Ultimately, by passing to the lim sup in the previous estimate, the strong convergence of $\{\mathbf{E}_k\}_{k\in\mathbb{N}_0}$ follows readily from Lemma 3.1 and (4.13). ∎

From Theorem 4.4, we can easily see the convergence of Algorithm 4.1 if we are able to prove that $\mathbf{E}_\infty$ is also the unique solution to (VI). To do so, let us first split each $\mathcal{T}_k$ as follows:

$$\mathcal{T}_k^+ := \bigcap_{l\geq k} \mathcal{T}_l, \quad \mathcal{T}_k^0 := \mathcal{T}_k \backslash \mathcal{T}_k^+, \quad \Omega_k^+ := \bigcup_{T\in\mathcal{T}_k^+} D_T, \quad \Omega_k^0 := \bigcup_{T\in\mathcal{T}_k^0} D_T.$$

That is, $\mathcal{T}_k^+$ consists of all elements that are not refined after the kth iteration, whereas elements in $\mathcal{T}_k^0$ are refined at least once after the kth iteration. Obviously, $\mathcal{T}_l^+ \subset \mathcal{T}_k^+$ for $l < k$, and we have $\mathcal{M}_k \subset \mathcal{T}_k^0$ for the set of the marked elements from Algorithm 4.1. Furthermore, we define a mesh-size function $h_k \colon \bar{\Omega} \to \mathbb{R}_+$ by $h_k(\boldsymbol{x}) = h_T$ for $\boldsymbol{x}$ in the interior of an element $T \in \mathcal{T}_k$ and $h_k(\boldsymbol{x}) = h_F$ for $\boldsymbol{x}$ in the relative interior of a face $F \in \mathcal{F}_k$. This mesh-size function has a property that is crucial for our further analysis (see [31, Corollary 4.1] and [36, Corollary 3.3]).

LEMMA 4.5. *Let χ_k^0 be the characteristic function of Ω_k^0. Then it holds that*

$$\lim_{k\to\infty} \|h_k \chi_k^0\|_{L^\infty(\Omega)} = 0.$$

In order to prove that the maximal error indicator in each loop of Algorithm 4.1 converges to zero, we need an additional assumption for the sequence of regularization parameters $\{\gamma_k\}_{k\in\mathbb{N}_0}$.

Assumption 4.6. There is a constant $C > 0$ independent of $k \in \mathbb{N}_0$ such that the sequence of Moreau–Yosida regularization parameters $\{\gamma_k\}_{k\in\mathbb{N}_0}$ satisfies

$$\gamma_k h_{\tilde{T}_k} \leq C \quad \forall k \in \mathbb{N}_0, \tag{4.15}$$

where $\tilde{T}_k$ denotes the element with the largest error estimator in the kth refinement step of Algorithm 4.1.

LEMMA 4.7. *Let Assumptions* 2.1 *and* 4.6 *hold, and let* $\{(\mathcal{T}_k, \mathcal{M}_k, \mathbf{E}_k, \boldsymbol{\lambda}_k)\}_{k\in\mathbb{N}_0}$ *be the sequence generated by Algorithm* 4.1. *Then it holds that*

$$\lim_{k\to\infty} \max_{T\in\mathcal{M}_k} \eta_k(\mathbf{E}_k, \boldsymbol{\lambda}_k, \mathbf{f}, T) = 0.$$

Proof. For convenience, we denote the element with the largest error indicator in $\mathcal{M}_k$ by $\tilde{T}_k$. Since $\tilde{T}_k \in \Omega_k^0$, the local quasi-uniformity and Lemma 4.5 imply that

$$|\omega_{\tilde{T}_k}| \leq C|\tilde{T}_k| \leq C\|h_k\chi_k^0\|^3_{L^\infty(\Omega)} \to 0 \quad \text{as } k\to\infty. \tag{4.16}$$

By using Lemma 4.2 and (3.3), (4.15), and (4.16), we obtain

$$\begin{aligned}
&\eta_k(\mathbf{E}_k, \boldsymbol{\lambda}_k, \mathbf{f}, j_c, \tilde{T}_k) \\
&\quad \leq C\left(\|\mathbf{E}_k\|_{0,\omega_{\tilde{T}_k}} + (1+\gamma_k h_{\tilde{T}_k})\|\,\mathbf{curl}\,\mathbf{E}_k\|_{0,\omega_{\tilde{T}_k}} + \|\boldsymbol{\lambda}_k\|_{0,\omega_{\tilde{T}_k}} + h_{\tilde{T}_k}\|\mathbf{f}\|_{0,\tilde{T}_k}\right) \\
&\quad \leq C\Big(\|\,\mathbf{curl}(\mathbf{E}_k - \mathbf{E}_\infty)\|_{0,\Omega} + \|\mathbf{E}_k - \mathbf{E}_\infty\|_{0,\Omega} + \|\,\mathbf{curl}\,\mathbf{E}_\infty\|_{0,\omega_{\tilde{T}_k}} \\
&\qquad + \|\mathbf{E}_\infty\|_{0,\omega_{\tilde{T}_k}} + h^{3/2}_{\tilde{T}_k}\|j_c\|_{L^\infty(\Omega)} + h_{\tilde{T}_k}\|\mathbf{f}\|_{0,\tilde{T}_k}\Big).
\end{aligned}$$

Now Theorem 4.4 readily implies the convergence of the first two terms, and (4.16) yields the convergence of the remaining terms, leading to the desired result. ∎

To proceed our analysis, we introduce the residual with respect to $\mathbf{E}_k \in \mathbf{V}_k$:

$$\langle \boldsymbol{\mathcal{R}}(\mathbf{E}_k), \mathbf{v}\rangle := a(\mathbf{E}_k, \mathbf{v}) + (\boldsymbol{\lambda}_k, \mathbf{v})_{0,\Omega} - (\mathbf{f}, \mathbf{v})_{0,\Omega} \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}), \tag{4.17}$$

which satisfies the Galerkin orthogonality (as $\mathbf{E}_k$ solves (3.3))

$$\langle \boldsymbol{\mathcal{R}}(\mathbf{E}_k), \mathbf{v}_k\rangle = 0 \quad \forall \mathbf{v}_k \in \mathbf{V}_k, k \in \mathbb{N}_0. \tag{4.18}$$

LEMMA 4.8. *Under Assumptions* 2.1 *and* 4.6*, the following convergence holds for the residual defined in* (4.17) *for the sequence* $\{\mathbf{E}_k\}_{k\in\mathbb{N}_0}$ *generated by Algorithm* 4.1*:*

$$\lim_{k\to\infty} \langle \boldsymbol{\mathcal{R}}(\mathbf{E}_k), \mathbf{v}\rangle = 0 \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}).$$

Proof. Let $\mathbf{v} \in \mathbf{C}_0^\infty(\Omega)$, and set $\mathbf{w} := \mathbf{v} - \boldsymbol{\Pi}_k \mathbf{v}$, where $\boldsymbol{\Pi}_k \colon \mathbf{H}_0(\mathbf{curl}) \to \mathbf{V}_k$ denotes the **curl**-conforming Nédélec interpolant [30]. By virtue of Lemma 3.5, there exist $\boldsymbol{\phi} \in \mathbf{H}_0^1(\Omega)$ and $\varphi \in H_0^1(\Omega)$ such that

$$\mathbf{w} - \boldsymbol{\Pi}_k^s \mathbf{w} = \boldsymbol{\phi} + \nabla\varphi.$$

Hence, the Galerkin orthogonality (4.18) yields

$$\begin{aligned} \langle \boldsymbol{\mathcal{R}}(\mathbf{E}_k), \mathbf{v}\rangle &= \langle \boldsymbol{\mathcal{R}}(\mathbf{E}_k), \mathbf{v} - \boldsymbol{\Pi}_k\mathbf{v}\rangle = \langle \boldsymbol{\mathcal{R}}(\mathbf{E}_k), \mathbf{w} - \boldsymbol{\Pi}_k^s\mathbf{w}\rangle \\ &= \langle \boldsymbol{\mathcal{R}}(\mathbf{E}_k), \boldsymbol{\phi}\rangle + \langle \boldsymbol{\mathcal{R}}(\mathbf{E}_k), \nabla\varphi\rangle. \end{aligned} \tag{4.19}$$

We will begin by estimating the first term on the right-hand side of (4.19). It follows by integration by parts, the trace theorem [38, page 87], and the stability estimate for $\boldsymbol{\phi}$ (cf. Lemma 3.5) that

$$\begin{aligned} \langle \boldsymbol{\mathcal{R}}(\mathbf{E}_k), \boldsymbol{\phi}\rangle &= -\sum_{T\in\mathcal{T}_k} (\mathbf{R}_T, \boldsymbol{\phi})_{0,T} + \sum_{F\in\mathcal{F}_k(\Omega)} (\mathbf{J}_{F,1}, \boldsymbol{\phi})_{0,F} \\ &\le \sum_{T\in\mathcal{T}_k} h_T \|\mathbf{R}_T\|_{0,T} h_T^{-1}\|\boldsymbol{\phi}\|_{0,T} + \sum_{F\in\mathcal{F}_k(\Omega)} h_F^{1/2}\|\mathbf{J}_{F,1}\|_{0,F} h_F^{-1/2}\|\boldsymbol{\phi}\|_{0,F} \\ &\le C \sum_{T\in\mathcal{T}_k} \eta_{k,1}(\mathbf{E}_k, \boldsymbol{\lambda}_k, \mathbf{f}, T)(h_T^{-1}\|\boldsymbol{\phi}\|_{0,T} + \|\nabla\boldsymbol{\phi}\|_{0,T}) \\ &\le C \sum_{T\in\mathcal{T}_k} \eta_{k,1}(\mathbf{E}_k, \boldsymbol{\lambda}_k, \mathbf{f}, T)\|\operatorname{\mathbf{curl}}(\mathbf{v} - \boldsymbol{\Pi}_k\mathbf{v})\|_{0,\tilde{D}_T}. \end{aligned}$$

To estimate the second term on the right-hand side of (4.19), we use similar arguments and the fact that $\operatorname{div} \epsilon\mathbf{E}_k|_T = 0$ in every $T \in \mathcal{T}_k$, the regularity of $\boldsymbol{\lambda}_k|_T \in \mathbf{H}(\operatorname{div}, T)$ from Lemma 3.3, as well as Lemma 3.5 to derive

$$\begin{aligned} \langle \boldsymbol{\mathcal{R}}(\mathbf{E}_k), \nabla\varphi\rangle &= (\epsilon\mathbf{E}_k + \boldsymbol{\lambda}_k, \nabla\varphi)_{0,\Omega} = -\sum_{T\in\mathcal{T}_k} (\operatorname{div}\boldsymbol{\lambda}_k, \varphi)_{0,T} + \sum_{F\in\mathcal{F}_k(\Omega)} (J_{F,2}, \varphi)_{0,F} \\ &\le C \sum_{T\in\mathcal{T}_k} h_T\|\operatorname{div}\boldsymbol{\lambda}_k\|_{0,T} h_T^{-1}\|\varphi\|_{0,T} + \sum_{F\in\mathcal{F}_k} h_F^{1/2}\|J_{F,2}\|_{0,F} h_F^{-1/2}\|\varphi\|_{0,F} \\ &\le C \sum_{T\in\mathcal{T}_k} \eta_{k,2}(\mathbf{E}_k, \boldsymbol{\lambda}_k, T)(h_T^{-1}\|\varphi\|_{0,T} + \|\nabla\varphi\|_{0,T}) \\ &\le C \sum_{T\in\mathcal{T}_k} \eta_{k,2}(\mathbf{E}_k, \boldsymbol{\lambda}_k, T)\|\mathbf{v} - \boldsymbol{\Pi}_k\mathbf{v}\|_{0,\tilde{D}_T}. \end{aligned}$$

Thus, combining the above estimates with (4.19) yields

$$|\langle \boldsymbol{\mathcal{R}}(\mathbf{E}_k), \mathbf{v}\rangle| \leq C \sum_{T\in\mathcal{T}_k} \eta_k(\mathbf{E}_k, \boldsymbol{\lambda}_k, \mathbf{f}, T)\|\mathbf{v} - \boldsymbol{\Pi}_k\mathbf{v}\|_{\mathbf{H}(\mathbf{curl},\tilde{D}_T)}. \tag{4.20}$$

Since the right-hand side of (4.20) depends on the enlarged element patch $\tilde{D}_T$, we introduce a buffer layer of elements between T_l and T_k for $k, l \in \mathbb{N}$ with $k > l$ by

$$\mathcal{T}_{k,l}^b \coloneqq \{T \in \mathcal{T}_k\backslash\mathcal{T}_l^+ \ : \ T \cap T' \neq \emptyset \quad \forall T' \in \mathcal{T}_l^+\}.$$

The uniform shape regularity of $\{\mathcal{T}_k\}_{k\in\mathbb{N}}$ and the fact that $\mathcal{T}_l^+ \subset \mathcal{T}_k^+ \subset \mathcal{T}_k$ yield

$$|\mathcal{T}_{k,l}^b| \leq C_l|\mathcal{T}_l^+| \tag{4.21}$$

with a constant $C_l > 0$ depending only on $\mathcal{T}_0$ and $\tilde{D}_T \subset \Omega_l^0$ for any $T \in \mathcal{T}_k\backslash(\mathcal{T}_l^+ \cup \mathcal{T}_{k,l}^b)$ (cf. [41]). We note that in this context, $|\mathcal{M}|$ denotes the number of elements contained in $\mathcal{M} \subset \mathcal{T}_k$. With these preparations, we can split $\mathcal{T}_k$ into $\mathcal{T}_l^+ \cup \mathcal{T}_{k,l}^b$ and $\mathcal{T}_k\backslash(\mathcal{T}_l^+ \cup \mathcal{T}_{k,l}^b)$ and derive by (4.20) that

$$\begin{aligned} |\langle \boldsymbol{\mathcal{R}}(\mathbf{E}_k), \mathbf{v}\rangle| \leq C\Big(&\eta_k(\mathbf{E}_k, \boldsymbol{\lambda}_k, \mathbf{f}, \mathcal{T}_k\backslash(\mathcal{T}_l^+ \cup \mathcal{T}_{k,l}^b))\|\mathbf{v} - \boldsymbol{\Pi}_k\mathbf{v}\|_{\mathbf{H}(\mathbf{curl},\Omega_l^0)} \\ &+ \eta_k(\mathbf{E}_k, \boldsymbol{\lambda}_k, \mathbf{f}, \mathcal{T}_l^+ \cup \mathcal{T}_{k,l}^b)\|\mathbf{v} - \boldsymbol{\Pi}_k\mathbf{v}\|_{\mathbf{H}(\mathbf{curl})}\Big). \end{aligned} \tag{4.22}$$

The stability estimate in Lemma 4.2 and Theorem 4.4 together with the error estimate for $\boldsymbol{\Pi}_k$ (see [30, Theorem 5.41]) yield

$$\eta_k(\mathbf{E}_k, \boldsymbol{\lambda}_k, \mathbf{f}, \mathcal{T}_k\backslash(\mathcal{T}_l^+ \cup \mathcal{T}_{k,l}^b))\|\mathbf{v} - \boldsymbol{\Pi}_k\mathbf{v}\|_{\mathbf{H}(\mathbf{curl},\Omega_l^0)} \leq C\|h_l\chi_l^0\|_{L^\infty(\Omega)}\|\mathbf{v}\|_{\mathbf{H}^2(\Omega)}. \tag{4.23}$$

As before, Lemma 4.5 ensures that (4.23) becomes small for a (fixed) sufficiently large $l \in \mathbb{N}$. Moreover, by using (4.1) and (4.21), we obtain

$$\begin{aligned} \eta_k(\mathbf{E}_k, \boldsymbol{\lambda}_k, \mathbf{f}, \mathcal{T}_l^+ \cup \mathcal{T}_{k,l}^b) &\leq \sqrt{|\mathcal{T}_l^+| + |\mathcal{T}_{k,l}^b|} \max_{T\in\mathcal{T}_l^+\cup\mathcal{T}_{k,l}^b} \eta_k(\mathbf{E}_k, \boldsymbol{\lambda}_k, \mathbf{f}, T) \\ &\leq \sqrt{(C_l + 1)|\mathcal{T}_l^+|} \max_{T\in\mathcal{M}_k} \eta_k(\mathbf{E}_k, \boldsymbol{\lambda}_k, \mathbf{f}, T). \end{aligned} \tag{4.24}$$

In view of Lemma 4.7, this gets smaller and smaller for increasing $k > k_0$ with a sufficiently large $k_0 \in \mathbb{N}$. Hence, we can combine (4.22)–(4.24) to obtain

$$\lim_{k\to\infty} \langle \boldsymbol{\mathcal{R}}(\mathbf{E}_k), \mathbf{v}\rangle = 0 \quad \forall \mathbf{v} \in \mathbf{C}_0^\infty(\Omega),$$

which completes the proof by exploiting the density of $\mathbf{C}_0^\infty(\Omega)$ in $\mathbf{H}_0(\mathbf{curl})$. ∎

With the help of Lemma 4.8, we are able to prove the following important result.

THEOREM 4.9. *Under Assumptions* 2.1 *and* 4.6*, the solution* $\mathbf{E}_\infty \in \mathbf{H}_0(\mathbf{curl})$ *of* (VI_∞) *solves* (VI)*, i.e.,*

$$a(\mathbf{E}_\infty, \mathbf{v} - \mathbf{E}_\infty) + \varphi(\mathbf{v}) - \varphi(\mathbf{E}_\infty) \geq \int_\Omega \mathbf{f} \cdot (\mathbf{v} - \mathbf{E}_\infty)\, d\boldsymbol{x} \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}).$$

Proof. Let $\mathbf{v} \in \mathbf{H}_0(\mathbf{curl})$. By virtue of (4.17), it holds for every $k \in \mathbb{N}$ that

$$a(\mathbf{E}_k, \mathbf{v} - \mathbf{E}_\infty) = \langle \boldsymbol{\mathcal{R}}(\mathbf{E}_k), \mathbf{v} - \mathbf{E}_\infty\rangle + (\mathbf{f}, \mathbf{v} - \mathbf{E}_\infty)_{0,\Omega} - (\boldsymbol{\lambda}_k, \mathbf{v} - \mathbf{E}_\infty)_{0,\Omega}\,,$$

from which we can derive

$$
\begin{aligned}
&a(\mathbf{E}_\infty, \mathbf{v}-\mathbf{E}_\infty)+\varphi(\mathbf{v})-\varphi(\mathbf{E}_\infty)\\
&\quad=\liminf_{k\to\infty}\left[a(\mathbf{E}_\infty-\mathbf{E}_k, \mathbf{v}-\mathbf{E}_\infty)+\varphi(\mathbf{v})-\varphi(\mathbf{E}_\infty)+a(\mathbf{E}_k, \mathbf{v}-\mathbf{E}_\infty)\right]\\
&\quad=\liminf_{k\to\infty}\left[a(\mathbf{E}_\infty-\mathbf{E}_k, \mathbf{v}-\mathbf{E}_\infty)+\langle\boldsymbol{\mathcal{R}}(\mathbf{E}_k), \mathbf{v}-\mathbf{E}_\infty\rangle\right.\\
&\qquad\qquad\left.+\varphi(\mathbf{v})-\varphi(\mathbf{E}_\infty)-(\boldsymbol{\lambda}_k, \mathbf{v}-\mathbf{E}_\infty)_{0,\Omega}\right]+(\mathbf{f}, \mathbf{v}-\mathbf{E}_\infty)_{0,\Omega}\\
&\quad\geq\liminf_{k\to\infty}\left[a(\mathbf{E}_\infty-\mathbf{E}_k, \mathbf{v}-\mathbf{E}_\infty)+\langle\boldsymbol{\mathcal{R}}(\mathbf{E}_k), \mathbf{v}-\mathbf{E}_\infty\rangle\right]\\
&\qquad\qquad+\liminf_{k\to\infty}\left[\varphi(\mathbf{v})-\varphi(\mathbf{E}_\infty)-(\boldsymbol{\lambda}_k, \mathbf{v}-\mathbf{E}_\infty)_{0,\Omega}\right]+(\mathbf{f}, \mathbf{v}-\mathbf{E}_\infty)_{0,\Omega}.
\end{aligned}
\tag{4.25}
$$

Using Theorem 4.4 and Lemma 4.8, we get the convergence of the first limit on the right-hand side of (4.25):

$$
\begin{aligned}
&|a(\mathbf{E}_\infty-\mathbf{E}_k, \mathbf{v}-\mathbf{E}_\infty)+\langle\boldsymbol{\mathcal{R}}(\mathbf{E}_k), \mathbf{v}-\mathbf{E}_\infty\rangle|\\
&\quad\leq\|\mathbf{E}_\infty-\mathbf{E}_k\|_a\|\mathbf{v}-\mathbf{E}_\infty\|_a+|\langle\boldsymbol{\mathcal{R}}(\mathbf{E}_k), \mathbf{v}-\mathbf{E}_\infty\rangle|\to 0 \text{ as } k\to\infty.
\end{aligned}
\tag{4.26}
$$

In order to estimate the remaining terms on the right-hand side of (4.25), we subtract (3.3) from (4.7) to obtain with $\mathbf{v}_\infty=\mathbf{v}_k=\mathbf{E}_k$ that

$$
a(\mathbf{E}_\infty-\mathbf{E}_k, \mathbf{E}_k)=\int_\Omega(\boldsymbol{\lambda}_k-\boldsymbol{\lambda}_\infty)\cdot\mathbf{E}_k\,d\boldsymbol{x}\quad\forall k\in\mathbb{N}.
\tag{4.27}
$$

Then all these remaining terms can be estimated (with $|\boldsymbol{\lambda}_k|\leq j_c$ a.e. in Ω) as follows:

$$
\begin{aligned}
&\liminf_{k\to\infty}\left[\varphi(\mathbf{v})-\varphi(\mathbf{E}_\infty)-(\boldsymbol{\lambda}_k, \mathbf{v})_{0,\Omega}+(\boldsymbol{\lambda}_k, \mathbf{E}_\infty)_{0,\Omega}\right]\\
&\quad\geq\liminf_{k\to\infty}(\boldsymbol{\lambda}_k, \mathbf{E}_\infty)_{0,\Omega}-\varphi(\mathbf{E}_\infty)\underbrace{=}_{(4.7)}\liminf_{k\to\infty}(\boldsymbol{\lambda}_k-\boldsymbol{\lambda}_\infty, \mathbf{E}_\infty)_{0,\Omega}\\
&\quad=\liminf_{k\to\infty}\left[(\boldsymbol{\lambda}_k-\boldsymbol{\lambda}_\infty, \mathbf{E}_\infty-\mathbf{E}_k)_{0,\Omega}+(\boldsymbol{\lambda}_k-\boldsymbol{\lambda}_\infty, \mathbf{E}_k)_{0,\Omega}\right]\\
&\quad\overbrace{=}^{(4.27)}\liminf_{k\to\infty}\left[(\boldsymbol{\lambda}_k-\boldsymbol{\lambda}_\infty, \mathbf{E}_\infty-\mathbf{E}_k)_{0,\Omega}+a(\mathbf{E}_\infty-\mathbf{E}_k, \mathbf{E}_k)\right]=0.
\end{aligned}
\tag{4.28}
$$

Finally, inserting (4.26) and (4.28) into (4.25) concludes that $\mathbf{E}_\infty\in\mathbf{H}_0(\mathbf{curl})$ is the unique solution to (VI). ∎

Theorems 4.4 and 4.9 lead to our main convergence result for Algorithm 4.1.

THEOREM 4.10. *Let Assumptions* 2.1 *and* 4.6 *hold. Furthermore, let* $\{\mathbf{E}_k\}_{k\in\mathbb{N}_0}$ *be the sequence generated by Algorithm* 4.1 *and* $\mathbf{E}\in\mathbf{H}_0(\mathbf{curl})$ *denote the solution to* (VI). *Then*

$$
\lim_{k\to\infty}\|\mathbf{E}_k-\mathbf{E}\|_{\mathbf{H}(\mathbf{curl})}=0\quad\textit{and}\quad\lim_{k\to\infty}\|\boldsymbol{\lambda}_k-\boldsymbol{\lambda}_\infty\|_{\mathbf{V}_k^*}=0
\tag{4.29}
$$

with the dual norm $\|\boldsymbol{\lambda}_k-\boldsymbol{\lambda}\|_{\mathbf{V}_k^*}:=\sup\left\{(\boldsymbol{\lambda}_k-\boldsymbol{\lambda}, \mathbf{v})_{0,\Omega}/\|\mathbf{v}\|_{\mathbf{H}(\mathbf{curl})}:\mathbf{v}\in\mathbf{V}_k\setminus\{0\}\right\}$ *for all* $k\in\mathbb{N}_0$.

Proof. The first convergence in (4.29) follows by combining Theorems 4.4 and 4.9. Furthermore, in view of (3.1) and (3.3), we have that

$$
\sup_{\mathbf{v}\in\mathbf{V}_k\setminus\{0\}}\frac{(\boldsymbol{\lambda}_k-\boldsymbol{\lambda}, \mathbf{v})_{0,\Omega}}{\|\mathbf{v}\|_{\mathbf{H}(\mathbf{curl})}}=\sup_{\mathbf{v}\in\mathbf{V}_k\setminus\{0\}}\frac{a(\mathbf{E}-\mathbf{E}_k, \mathbf{v})}{\|\mathbf{v}\|_{\mathbf{H}(\mathbf{curl})}}\leq 2\max\{\overline{\epsilon}, \underline{\mu}^{-1}\}\|\mathbf{E}-\mathbf{E}_k\|_{\mathbf{H}(\mathbf{curl})},
$$

which implies the second convergence in (4.29) and concludes the proof. ∎

5. Numerical results. As pointed out in the introduction, we apply Algorithm 4.1 to confirm a physical phenomenon numerically from the type-II (high-temperature) superconductivity. Type-II superconductors are characterized by the loss of electrical resistance and the repulsion of weak magnetic fields when cooled down below a certain critical temperature. The latter is known as the Meissner effect. In order to describe the penetration and exit of the magnetic flux in a type-II superconductor, Bean [4, 5] proposed a critical state model that relates the electrical field $\mathbf{E}$ and the current density $\mathbf{J}$. By combining his model with Maxwell's equations, we obtain a nonlinear hyperbolic mixed Maxwell system which is equivalent to a hyperbolic mixed VI of second kind [44]. We can deduce a stationary mixed problem satisfied by the electromagnetic field by considering, e.g., a semidiscretization in time, i.e.,

$$\begin{cases} \int_\Omega \epsilon \mathbf{E}\cdot(\mathbf{v}-\mathbf{E}) + \mu^{-1}\mathbf{B}\cdot(\mathbf{w}-\mathbf{B})\,d\boldsymbol{x} \\ \quad + \int_\Omega \mu^{-1}\,\mathbf{curl}\,\mathbf{E}\cdot\mathbf{w} - \mu^{-1}\mathbf{B}\cdot\mathbf{curl}\,\mathbf{v}\,d\boldsymbol{x} \\ \quad + \varphi(\mathbf{v}) - \varphi(\mathbf{E}) \geq \int_\Omega \mathbf{f}\cdot(\mathbf{v}-\mathbf{E})\,d\boldsymbol{x} \\ \quad \text{for all } (\mathbf{v},\mathbf{w}) \in \mathbf{H}_0(\mathbf{curl}) \times \mathbf{L}^2(\Omega), \end{cases} \tag{5.1}$$

where ϵ, μ and $\mathbf{f}$ satisfy Assumption 2.1. In (5.1), $\mathbf{E}$ denotes the electric field, $\mathbf{B}$ stands for the magnetic induction, and the right-hand side $\mathbf{f}$ is the applied current source. Moreover, ϵ and μ represent the electric permittivity and the magnetic permeability, respectively.

Now we are able to decouple (5.1) by taking $\mathbf{v} = \mathbf{E}$. Hence, we have $\mathbf{B} = -\,\mathbf{curl}\,\mathbf{E}$. Using this and taking $\mathbf{w} = \mathbf{B}$ in (5.1), we obtain

$$a(\mathbf{E},\mathbf{v}-\mathbf{E}) + \varphi(\mathbf{v}) - \varphi(\mathbf{E}) \geq \int_\Omega \mathbf{f}\cdot(\mathbf{v}-\mathbf{E})\,d\boldsymbol{x} \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}), \tag{5.2}$$

which corresponds to the variational inequality (VI) of our interest. Then we can use Algorithm 4.1 to compute the electrical field $\mathbf{E}$ from (5.2) and get the magnetic induction from $\mathbf{B} = -\,\mathbf{curl}\,\mathbf{E}$.

Let us now specify the numerical setup. We choose the computational domain $\Omega = (-2, 2)^3$ and the right-hand side $\mathbf{f} \in \mathbf{L}^2(\Omega)$ that satisfies the divergence-free condition (A2) as a circular current applied to a pipe coil $\Omega_p \subset \Omega$ with inner radius $r_p = 0.8$, e.g.,

$$\mathbf{f}(x_1,x_2,x_3) = \begin{cases} 1/R\left(0,\ -x_3/(x_2^2+x_3^2)^{1/2},\ x_2/(x_2^2+x_3^2)^{1/2}\right) & \text{for } (x_1,x_2,x_3) \in \Omega_p, \\ 0 & \text{for } (x_1,x_2,x_3) \notin \Omega_p, \end{cases}$$

where the constant $R > 0$ denotes the electrical resistance of the pipe coil Ω_p and is set to be $R = 10^3$ in our experiments. The superconductor $\Omega_{sc} \subset \Omega$ is a ball around the origin with the radius $r_{sc} = 0.5$. The physical parameters ϵ and μ are both taken to be 1. All implementations were realized with the open-source finite-element computational software FEniCS [29], and we used Paraview to visualize the numerical outcome. All the figures presented in this section are the two-dimensional (2D) slices of the original three-dimensional plots.

We initialize Algorithm 4.1 with a coarse uniform mesh $\mathcal{T}_0$ consisting of 384 cells and set

$$\gamma_k(\mathcal{T}_k) = \sqrt{|\mathcal{T}_k|} + \gamma_0, \quad \gamma_0 = 7 \times 10^4,$$

TABLE 1
The degrees of freedom and a numerical verification of Assumption 4.6 (first example).

k	2	3	4	5	6	7	8	9
$\gamma_k h_{\tilde{T}_k}$	6.065e+4	4.953e+4	2.478e+4	1.518e+4	1.754e+4	6.213e+3	3.817e+3	1.920e+3
#DoFs	1236	2436	7486	14521	32837	97840	315675	1068215

TABLE 2
The degrees of freedom and a numerical verification of Assumption 4.6 (second example).

k	1	2	3	4	5	6	7	8
$\gamma_k h_{\tilde{T}_k}$	1.213e+5	6.065e+4	3.917e+4	4.958e+4	2.481e+4	1.758e+4	1.248e+4	1.930e+3
#DoFs	604	1670	5754	14882	35824	114620	443502	2045523

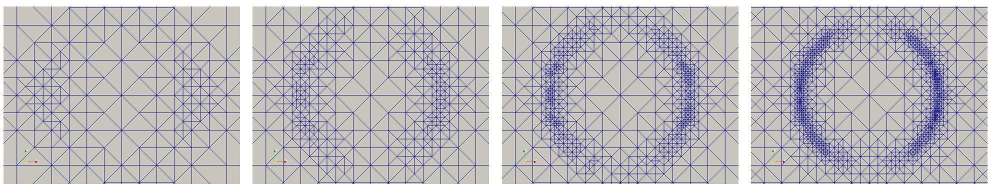

FIG. 1. *Evolution of the adaptive mesh (2D slice) for the first example in steps* $k = 6, 7, 8, 9$.

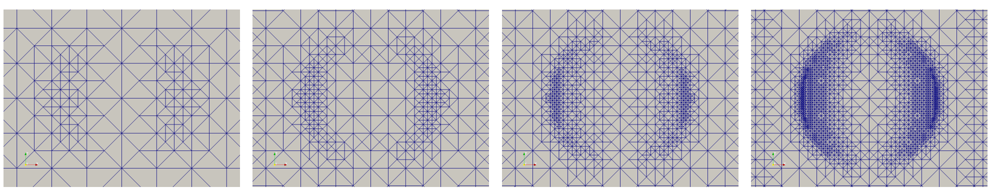

FIG. 2. *Evolution of the adaptive mesh (2D slice) for the second example in steps* $k = 5, 6, 7, 8$.

which apparently satisfies (4.2). Moreover, to verify Assumption 4.6 numerically, we computed the factor $\gamma_k h_{\tilde{T}_k}$ in every step of Algorithm 4.1 (see Tables 1 and 2). Our numerical results confirmed Assumption 4.6 for the above choice of γ_k. Indeed, it holds that

$$\gamma_k h_{\tilde{T}_k} \leq \gamma_1 h_{\tilde{T}_1},$$

for all iterations k. The very complicated structure of (VI) makes it practically impossible to find an analytical solution. Thus, we tested Algorithm 4.1 in a setup where we would expect a certain behavior. For this purpose, we selected the critical current density $j_c(\boldsymbol{x}) = 0.1\chi_{\Omega_{sc}}(\boldsymbol{x})$, where we know the interface between the superconducting and the normal region a priori—due to the comparatively high critical current density, there is no penetration of the superconductor at all. Hence, the interface corresponds to the surface of the superconductor Ω_{sc}. In Figure 1 the evolution of the adaptive mesh is shown, and the corresponding magnetic field lines are displayed in Figure 4(a). Our expectations are confirmed since we can observe that Algorithm 4.1 adaptively refines the mesh around the surface of the superconductor and that there is no magnetic field penetration.

Keeping the observations of the first example in mind, we choose a significantly smaller critical current density $j_c(\boldsymbol{x}) = 0.001\chi_{\Omega_{sc}}(\boldsymbol{x})$ as the second example. The remaining parameters in the setup remain the same. In this case, we do not have

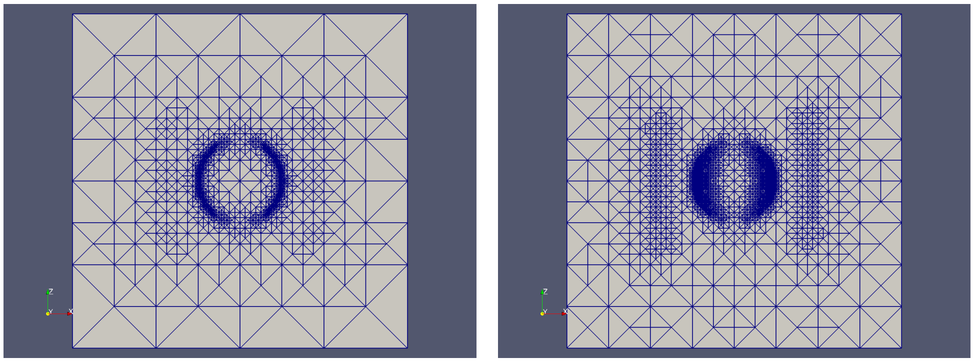

FIG. 3. *2D slices of the final meshes in total.*

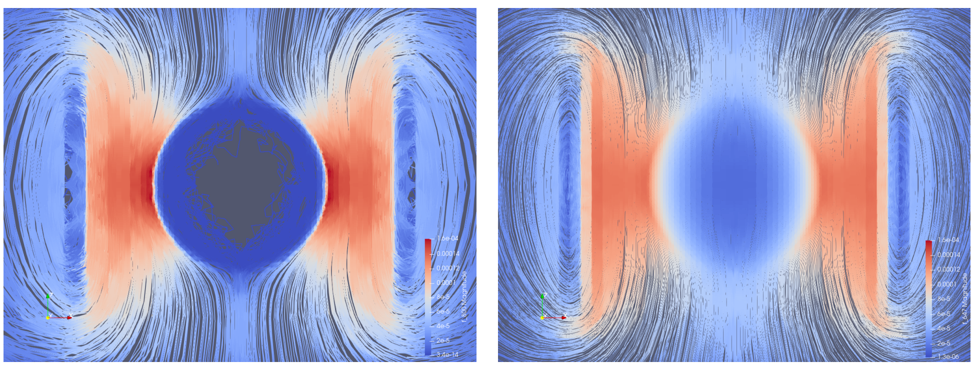

FIG. 4. *2D slices of the magnetic field lines.*

any (a priori) knowledge of the approximate position and shape of the mentioned interface. Therefore, this example is much more challenging, and the adaptivity is necessary to extract this interface. Again, Algorithm 4.1 adaptively refines the mesh, but it exhibits a wider interface than the first example, which indicates that the superconductor is partially penetrated by the magnetic field lines (see Figure 2). This is also confirmed by the magnetic field shown in Figure 4(b). Last but not least, we compare the final meshes in Figure 3 and observe that in both the examples the coil Ω_p is also refined. We justify this behavior with the fact that there is also a strong change of the magnetic field strength around Ω_p (see also Figure 4).

5.1. Convergence rate tests. We close our paper by reporting the numerical convergence order of Algorithm 4.1 for our numerical example. Since we do not know the exact solution, we consider a reference solution $\mathbf{E}_{ref}$ at a very fine adaptive mesh and test the convergence behavior of the adaptive solutions $\mathbf{E}_k$ toward the reference one. More precisely, this can be quantitatively clarified by evaluating the experimental rate of convergence (ERC) using two consecutive adaptive solutions and #DoFs:

$$\mathrm{ERC}_k = \left| \frac{\log(\|\mathbf{E}_k - \mathbf{E}_{ref}\|_{\boldsymbol{H}(\mathbf{curl})}) - \log(\|\mathbf{E}_{k-1} - \mathbf{E}_{ref}\|_{\boldsymbol{H}(\mathbf{curl})})}{\log(\#\mathrm{DoFs}_k) - \log(\#\mathrm{DoFs}_{k-1})} \right|.$$

For our two examples, we computed the experimental rate of convergence at different levels k and conclude a convergence order of around 0.15 for the first example and 0.35

Table 3
Experimental rates of convergence for the first example (left) and the second example (right).

k	4	5	8
ERC_k	0.1445	0.1558	0.1832

k	3	5	8
ERC_k	0.3288	0.3698	0.3663

for the second one (see Table 3). The very sharp phase transition in the first example implies that the corresponding solution is discontinuous around Ω_{sc} (see Figure 4(a)). On the other hand, for the second example, this transition is significantly smoother (see Figure 4(b)). This fact may explain the difference in the convergence rates.

REFERENCES

[1] M. Ainsworth and J. T. Oden, *A posteriori error estimation in finite element analysis*, Comput. Methods Appl. Mech. Engrg., 142 (1997), pp. 1–88.

[2] I. Babuška and W. Rheinboldt, *Error estimates for adaptive finite element computations*, SIAM J. Numer. Anal., 15 (1978), pp. 736–754.

[3] I. Babuška and M. Vogelius, *Feedback and adaptive finite element solution of one-dimensional boundary value problems*, Numer. Math., 44 (1984), pp. 75–102.

[4] C. P. Bean, *Magnetization of hard superconductors*, Phys. Rev. Lett., 8 (1962), pp. 250–253.

[5] C. P. Bean, *Magnetization of high-field superconductors*, Rev. Mod. Phys., 36 (1964), pp. 31–39.

[6] R. Beck, R. Hiptmair, R. H. W. Hoppe, and B. Wohlmuth, *Residual based a posteriori error estimators for eddy current computation*, ESAIM Math. Model. Numer. Anal., 34 (2000), pp. 159–182.

[7] V. Bostan, W. Han, and B. D. Reddy, *A posteriori error estimation and adaptive solution of elliptic variational inequalities of the second kind*, Appl. Numer. Math., 52 (2005), pp. 13–38.

[8] D. Braess, *A posteriori error estimators for obstacle problems—Another look*, Numer. Math., 101 (2005), pp. 415–421.

[9] D. Braess, C. Carstensen, and R. H. W. Hoppe, *Convergence analysis of a conforming adaptive finite element method for an obstacle problem*, Numer. Math., 107 (2007), pp. 455–471.

[10] Z. Cai and S. Cao, *A recovery-based a posteriori error estimator for H(curl) interface problems*, Comput. Methods Appl. Mech. Engrg., 296 (2015), pp. 169–195.

[11] Z. Cai, S. Cao, and R. Falgout, *Robust a posteriori error estimation for finite element approximation to H(curl) problem*, Comput. Methods Appl. Mech. Engrg., 309 (2016), pp. 182–201.

[12] C. Carstensen and J. Hu, *An optimal adaptive finite element method for an obstacle problem*, Comput. Methods Appl. Math., 15 (2015), pp. 259–277.

[13] J. Chen, Y. Xu, and J. Zou, *Convergence analysis of an adaptive edge element method for Maxwell's equations*, Appl. Numer. Math., 59 (2009), pp. 2950–2969.

[14] Z. Chen and R. H. Nochetto, *Residual type a posteriori error estimates for elliptic obstacle problems*, Numer. Math., 84 (2000), pp. 527–548.

[15] Z. Chen, L. Wang, and W. Zheng, *An adaptive multilevel method for time-harmonic Maxwell equations with singularities*, SIAM J. Sci. Comput., 29 (2007), pp. 118–138.

[16] P. Ciarlet, Jr., *On the approximation of electromagnetic fields by edge finite elements. Part 1: Sharp interpolation results for low-regularity fields*, Comput. Math. Appl., 71 (2016), pp. 85–104.

[17] M. Costabel and M. Dauge, *Singularities of electromagnetic fields in polyhedral domains*, Arch. Ration. Mech. Anal., 151 (2000), pp. 221–276.

[18] M. Costabel, M. Dauge, and S. Nicaise, *Singularities of Maxwell interface problems*, ESAIM Math. Model. Numer. Anal., 33 (1999), pp. 627–649.

[19] J. C. De Los Reyes and M. Hintermüller, *A duality based semismooth Newton framework for solving variational inequalities of the second kind*, IFB, 13 (2011), pp. 437–462.

[20] W. Dörfler, *A convergent adaptive algorithm for Poisson's equation*, SIAM J. Numer. Anal., 33 (1996), pp. 1106–1124.

[21] K. Eriksson and C. Johnson, *Adaptive finite element methods for parabolic problems* I: *A linear model problem*, SIAM J. Numer. Anal., 28 (1991), pp. 43–77.

[22] R. Glowinski, J. L. Lions, and R. Trémolières, *Numerical Analysis of Variational Inequalities*, Studies in Mathematics and Its Applications, Elsevier Science, Amsterdam, 1981.
[23] R. H. W. Hoppe and R. Kornhuber, *Adaptive multilevel methods for obstacle problems*, SIAM J. Numer. Anal., 31 (1994), pp. 301–323.
[24] R. H. W. Hoppe and J. Schöberl, *Convergence of adaptive edge element methods for the 3d eddy currents equations*, J. Comput. Math., 27 (2009), pp. 657–676.
[25] K. Ito and K. Kunisch, *Lagrange Multiplier Approach to Variational Problems and Applications*, SIAM, Philadelphia, 2008.
[26] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, SIAM, Philadelphia, 2000.
[27] R. Kornhuber, *A posteriori error estimates for elliptic variational inequalities*, Comput. Math. Appl., 31 (1996), pp. 49–60.
[28] J. L. Lions and G. Stampacchia, *Variational inequalities*, Comm. Pure Appl. Math., 20 (1967), pp. 493–519.
[29] A. Logg, K.-A. Mardal, G. N. Wells, et al., *Automated Solution of Differential Equations by the Finite Element Method*, Springer, Berlin, 2012.
[30] P. Monk, *Finite Element Methods for Maxwell's Equations*, Numerical Analysis and Scientific Computation, Clarendon Press, Oxford, 2003.
[31] P. Morin, K. G. Siebert, and A. Veeser, *A basic convergence result for conforming adaptive finite elements*, Math. Models Methods Appl. Sci., 18 (2008), pp. 707–737.
[32] J.-C. Nédélec, *Mixed finite elements in* $\mathbb{R}^3$, Numer. Math., 35 (1980), pp. 315–341.
[33] R. H. Nochetto, K. G. Siebert, and A. Veeser, *Pointwise a posteriori error control for elliptic obstacle problems*, Numer. Math., 95 (2003), pp. 163–195.
[34] T. Roubícek, *Nonlinear Partial Differential Equations with Applications*, Internat. Ser. Numer. Math. 153, Springer, Basel, Switzerland, 2013.
[35] J. Schöberl, *A posteriori error estimates for Maxwell equations*, Math. Comp., 77 (2007), pp. 633–649.
[36] K. G. Siebert, *A convergence proof for adaptive finite elements without lower bound*, IMA J. Numer. Anal., 31 (2011), pp. 947–970.
[37] C. T. Traxler, *An algorithm for adaptive mesh refinement in n dimensions*, Computing, 59 (1997), pp. 115–137.
[38] R. Verfürth, *A Posteriori Error Estimation Techniques for Finite Element Methods*, Oxford University Press, Oxford, 2013.
[39] F. Wang and W. Han, *Another view for a posteriori error estimates for variational inequalities of the second kind*, Appl. Numer. Math., 72 (2013), pp. 225–233.
[40] M. Winckler and I. Yousept, *Fully discrete scheme for Bean's critical-state model with temperature effects in superconductivity*, SIAM J. Numer. Anal., 57 (2019), pp. 2685–2706.
[41] Y. Xu and J. Zou, *A convergent adaptive edge element method for an optimal control problem in magnetostatics*, ESAIM Math. Model. Numer. Anal., 51 (2017), pp. 615–640.
[42] I. Yousept, *Hyperbolic Maxwell variational inequalities of the second kind*, ESAIM Control Optim. Calc. Var., 26 (2020), to appear, https://doi.org/10.1051/cocv/2019015.
[43] I. Yousept, *Optimal control of quasilinear* $H(curl)$*-elliptic partial differential equations in magnetostatic field problems*, SIAM J. Control Optim., 51 (2013), pp. 3624–3651.
[44] I. Yousept, *Hyperbolic Maxwell variational inequalities for Bean's critical-state model in type-*II *superconductivity*, SIAM J. Numer. Anal., 55 (2017), pp. 2444–2464.
[45] I. Yousept, *Optimal control of non-smooth hyperbolic evolution Maxwell equations in type-*II *superconductivity*, SIAM J. Control Optim., 55 (2017), pp. 2305–2332.
[46] I. Yousept and J. Zou, *Edge element method for optimal control of stationary Maxwell system with Gauss law*, SIAM J. Numer. Anal., 55 (2017), pp. 2787–2810.