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On Maxwell's and Poincaré's Constants

by

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`Dedicated to Sergey Igorevich Repin on the occasion of his 60th birthday`

Abstract

We prove that for bounded and convex domains in three dimensions, the Maxwell constants are bounded from below and above by Friedrichs' and Poincaré's constants. In other words, the second Maxwell eigenvalues lie between the square roots of the second Neumann-Laplace and the first Dirichlet-Laplace eigenvalue.

Key Words Maxwell's equations, Maxwell constant, second Maxwell eigenvalue, electro statics, magneto statics, Poincaré's inequality, Friedrichs' inequality, Poincaré's constant, Friedrichs' constant

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1 Introduction

It is well known that, e.g., for bounded Lipschitz domains $\Omega \subset \mathbb{R}^3$, a square integrable vector field v having square integrable divergence $\operatorname{div} v$ and square integrable rotation vector field $\operatorname{rot} v$ as well as vanishing tangential or normal component on the boundary Γ, i.e, $v_{\mathtt{t}}|_{\Gamma} = 0$ resp. $v_{\mathtt{n}}|_{\Gamma} = 0$, satisfies the Maxwell estimate

$$\int_{\Omega} |v|^2 \leq c_{\mathtt{m}}^2 \int_{\Omega} \big(|\operatorname{rot} v|^2 + |\operatorname{div} v|^2 \big), \tag{1.1}$$

if in addition v is perpendicular to the so called Dirichlet or Neumann fields, i.e.,

$$\int_{\Omega} v \cdot w = 0 \quad \forall\, w \in \mathcal{H}(\Omega),$$

where

$$\mathcal{H}(\Omega) = \begin{cases} \mathcal{H}_{\mathtt{D}}(\Omega) := \{w \in \mathsf{L}^2(\Omega) \,:\, \operatorname{rot} w = 0,\, \operatorname{div} w = 0,\, w_{\mathtt{t}}|_\Gamma = 0\}, & \text{if } v_{\mathtt{t}}|_\Gamma = 0, \\ \mathcal{H}_{\mathtt{N}}(\Omega) := \{w \in \mathsf{L}^2(\Omega) \,:\, \operatorname{rot} w = 0,\, \operatorname{div} w = 0,\, w_{\mathtt{n}}|_\Gamma = 0\}, & \text{if } v_{\mathtt{n}}|_\Gamma = 0 \end{cases}$$

holds. Here, $c_{\mathtt{m}}$ is a positive constant independent of v, which will be called Maxwell constant. See, e.g., [20, 21, 13, 26]. We note that (1.1) is valid in much more general situations modulo some more or less obvious modifications, such as for mixed boundary conditions, in unbounded like exterior domains, in domains $\Omega \subset \mathbb{R}^N$, on N-dimensional Riemannian manifolds, for differential forms or in the case of inhomogeneous media. See, e.g.,[10, 15, 17, 21, 22, 23, 26, 27].

So far, to the best of the author's knowledge, general bounds for the Maxwell constants $c_{\mathtt{m}}$ are unknown. On the other hand, at least estimates for $c_{\mathtt{m}}$ from above are very important from the point of view of applications, such as preconditioning or a priori and a posteriori error estimation for numerical methods.

In this contribution we will prove that for bounded and convex domains $\Omega \subset \mathbb{R}^3$

$$c_{\mathtt{p},\circ} \leq c_{\mathtt{m}} \leq c_{\mathtt{p}} \leq \operatorname{diam}(\Omega)/\pi \tag{1.2}$$

holds true, where $0 < c_{\mathtt{p},\circ} < c_{\mathtt{p}}$ are the Poincaré constants, such that for all square integrable functions u having square integrable gradient ∇u

$$\int_\Omega |u|^2 \leq c_{\mathtt{p},\circ}^2 \int_\Omega |\nabla u|^2 \quad \text{resp.} \quad \int_\Omega |u|^2 \leq c_{\mathtt{p}}^2 \int_\Omega |\nabla u|^2$$

holds, if $u|_\Gamma = 0$ resp. $\int_\Omega u = 0$. While the result (1.2) is already well known in two dimensions, even for general Lipschitz domains $\Omega \subset \mathbb{R}^2$ (except of the last inequality), it is new in three dimensions. We note that the last inequality in (1.2) has been proved in the famous paper of Payne and Weinberger [19], where also the optimality of the estimate was shown. A small mistake in this paper has been corrected later in [3]. We will prove the crucial and from the applicational point of view most interesting inequality $c_{\mathtt{m}} \leq c_{\mathtt{p}}$ also for polyhedral domains in $\mathbb{R}^3$, which might not be convex but still allow the $\mathsf{H}^1(\Omega)$-regularity for solutions of Maxwell's equations. We will give a general result for non-smooth and inhomogeneous, anisotropic media as well. Let us note that our methods are only based on elementary calculations.

2 Preliminaries

Throughout this paper let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Many of our results hold true under weaker assumptions on the regularity of the boundary $\Gamma := \partial\Omega$. Essentially we need the compact embeddings (2.3)-(2.5) to hold. We will use the standard Lebesgue spaces $\mathsf{L}^2(\Omega)$ of square integrable functions or vector (or even tensor) fields equipped with the usual $\mathsf{L}^2(\Omega)$-scalar product $\langle\,\cdot\,,\,\cdot\,\rangle_\Omega$ and $\mathsf{L}^2(\Omega)$-norm $|\,\cdot\,|_\Omega$. Moreover, we will work with the standard $\mathsf{L}^2(\Omega)$-Sobolev spaces for the gradient grad $= \nabla$, the rotation rot $= \nabla\times$ and the divergence div $= \nabla\cdot$ denoted by

$$\begin{aligned}
\mathsf{H}^1(\Omega) &:= \mathsf{H}(\operatorname{grad};\Omega), & \mathring{\mathsf{H}}^1(\Omega) &:= \mathring{\mathsf{H}}(\operatorname{grad};\Omega) := \overline{\mathring{\mathsf{C}}^\infty(\Omega)}^{\mathsf{H}^1(\Omega)}, \\
\mathsf{D}(\Omega) &:= \mathsf{H}(\operatorname{div};\Omega), & \mathring{\mathsf{D}}(\Omega) &:= \mathring{\mathsf{H}}(\operatorname{div};\Omega) := \overline{\mathring{\mathsf{C}}^\infty(\Omega)}^{\mathsf{D}(\Omega)}, \\
\mathsf{R}(\Omega) &:= \mathsf{H}(\operatorname{rot};\Omega), & \mathring{\mathsf{R}}(\Omega) &:= \mathring{\mathsf{H}}(\operatorname{rot};\Omega) := \overline{\mathring{\mathsf{C}}^\infty(\Omega)}^{\mathsf{R}(\Omega)}.
\end{aligned}$$

In the latter three Hilbert spaces the classical homogeneous scalar, normal and tangential boundary traces are generalized, respectively. An index zero at the lower right corner of the latter spaces indicates a vanishing derivative, e.g.,

$$\mathring{\mathsf{R}}_0(\Omega) := \{E \in \mathring{\mathsf{R}}(\Omega) \,:\, \operatorname{rot} E = 0\}, \quad \mathsf{D}_0(\Omega) := \{E \in \mathsf{D}(\Omega) \,:\, \operatorname{div} E = 0\}.$$

Moreover, we introduce a symmetric, bounded (L^∞) and uniformly positive definite matrix field $\varepsilon : \Omega \to \mathbb{R}^{3\times 3}$ and the spaces of (harmonic) Dirichlet and Neumann fields

$$\mathcal{H}_{\mathtt{D},\varepsilon}(\Omega) := \mathring{\mathsf{R}}_0(\Omega) \cap \varepsilon^{-1}\mathsf{D}_0(\Omega), \quad \mathcal{H}_{\mathtt{N},\varepsilon}(\Omega) := \mathsf{R}_0(\Omega) \cap \varepsilon^{-1}\mathring{\mathsf{D}}_0(\Omega).$$

We will also use the weighted ε-$\mathsf{L}^2(\Omega)$-scalar product $\langle\,\cdot\,,\,\cdot\,\rangle_{\Omega,\varepsilon} := \langle\varepsilon\,\cdot\,,\,\cdot\,\rangle_\Omega$ and the corresponding induced weighted ε-$\mathsf{L}^2(\Omega)$-norm $|\,\cdot\,|_{\Omega,\varepsilon} := \langle\,\cdot\,,\,\cdot\,\rangle_{\Omega,\varepsilon}^{1/2}$. Moreover, $\perp_\varepsilon$ denotes orthogonality with respect to the ε-$\mathsf{L}^2(\Omega)$-scalar product. If we equip $\mathsf{L}^2(\Omega)$ with this weighted scalar product we write $\mathsf{L}^2_\varepsilon(\Omega)$. If ε equals the identity id, we skip it in our notations, e.g., we write $\perp := \perp_{\mathrm{id}}$ and $\mathcal{H}_{\mathtt{D}}(\Omega) := \mathcal{H}_{\mathtt{D},\mathrm{id}}(\Omega)$. By the assumptions on ε we have

$$\exists\, \underline{\varepsilon}, \overline{\varepsilon} > 0 \quad \forall\, E \in \mathsf{L}^2(\Omega) \quad \underline{\varepsilon}^{-2}|E|^2_\Omega \leq \langle\varepsilon E, E\rangle_\Omega \leq \overline{\varepsilon}^2|E|^2_\Omega \tag{2.1}$$

and we note $|E|^2_{\Omega,\varepsilon} = \langle\varepsilon E, E\rangle_\Omega = |\varepsilon^{1/2}E|^2_\Omega$ as well as $|\varepsilon E|_\Omega = |\varepsilon^{1/2}E|_{\Omega,\varepsilon}$. Thus, for all $E \in \mathsf{L}^2(\Omega)$

$$\underline{\varepsilon}^{-1}|E|_\Omega \leq |E|_{\Omega,\varepsilon} \leq \overline{\varepsilon}|E|_\Omega, \qquad \underline{\varepsilon}^{-1}|E|_{\Omega,\varepsilon} \leq |\varepsilon E|_\Omega \leq \overline{\varepsilon}|E|_{\Omega,\varepsilon}. \tag{2.2}$$

For later purposes let us also define $\hat{\varepsilon} := \max\{\underline{\varepsilon}, \overline{\varepsilon}\}$.

We have the following compact embeddings:

$$\mathring{\mathsf{H}}^1(\Omega) \subset \mathsf{H}^1(\Omega) \hookrightarrow \mathsf{L}^2(\Omega) \qquad \text{(Rellich's selection theorem)} \tag{2.3}$$

$$\mathring{\mathsf{R}}(\Omega) \cap \varepsilon^{-1}\mathsf{D}(\Omega) \hookrightarrow \mathsf{L}^2(\Omega) \qquad \text{(tangential Maxwell compactness property)} \tag{2.4}$$

$$\mathsf{R}(\Omega) \cap \varepsilon^{-1}\mathring{\mathsf{D}}(\Omega) \hookrightarrow \mathsf{L}^2(\Omega) \qquad \text{(normal Maxwell compactness property)} \tag{2.5}$$

It is well known and easy to prove by standard indirect arguments that (2.3) implies the Poincaré estimates

$$\exists\, c_{\mathrm{p},\circ} > 0 \qquad \forall\, u \in \mathring{\mathsf{H}}^1(\Omega) \qquad |u|_\Omega \leq c_{\mathrm{p},\circ}|\nabla u|_\Omega, \tag{2.6}$$

$$\exists\, c_{\mathrm{p}} > 0 \qquad \forall\, u \in \mathsf{H}^1(\Omega) \cap \mathbb{R}^\perp \qquad |u|_\Omega \leq c_{\mathrm{p}}|\nabla u|_\Omega. \tag{2.7}$$

Analogously, (2.4) implies $\dim \mathcal{H}_{\mathtt{D},\varepsilon}(\Omega) < \infty$[*], since the unit ball in $\mathcal{H}_{\mathtt{D},\varepsilon}(\Omega)$ is compact, and the tangential Maxwell estimate, i.e., there exists $c_{\mathtt{m},\mathtt{t},\varepsilon} > 0$ such that

$$\forall\, E \in \mathring{\mathsf{R}}(\Omega) \cap \varepsilon^{-1}\mathsf{D}(\Omega) \qquad |(1-\pi_{\mathtt{D}})E|_{\Omega,\varepsilon} \leq c_{\mathtt{m},\mathtt{t},\varepsilon}\big(|\operatorname{rot} E|_{\Omega}^{2} + |\operatorname{div} \varepsilon E|_{\Omega}^{2}\big)^{1/2}, \tag{2.8}$$

where $\pi_{\mathtt{D}} : \mathsf{L}^2_{\varepsilon}(\Omega) \to \mathcal{H}_{\mathtt{D},\varepsilon}(\Omega)$ denotes the ε-$\mathsf{L}^2(\Omega)$-orthogonal projector onto Dirichlet fields. Similar results hold if one replaces the tangential or electric boundary condition by the normal or magnetic one. More precisely, (2.5) implies $\dim \mathcal{H}_{\mathtt{N},\varepsilon}(\Omega) < \infty$ and the corresponding normal Maxwell estimate, i.e., there exists $c_{\mathtt{m},\mathtt{n},\varepsilon} > 0$ such that

$$\forall\, H \in \mathsf{R}(\Omega) \cap \varepsilon^{-1}\mathring{\mathsf{D}}(\Omega) \qquad |H - \pi_{\mathtt{N}} H|_{\Omega,\varepsilon} \leq c_{\mathtt{m},\mathtt{n},\varepsilon}\big(|\operatorname{rot} H|_{\Omega}^{2} + |\operatorname{div} \varepsilon H|_{\Omega}^{2}\big)^{1/2}, \tag{2.9}$$

where $\pi_{\mathtt{N}} : \mathsf{L}^2_{\varepsilon}(\Omega) \to \mathcal{H}_{\mathtt{N},\varepsilon}(\Omega)$ denotes the ε-$\mathsf{L}^2(\Omega)$-orthogonal projector onto Neumann fields. We note that $\sqrt{c_{\mathtt{m},\mathtt{t},\varepsilon}^2 + 1}$ can also be seen as the norm of the inverse M^{-1} of the corresponding electro static Maxwell operator

$$M \;:\; \begin{array}{ccc} \mathring{\mathsf{R}}(\Omega) \cap \varepsilon^{-1}\mathsf{D}(\Omega) \cap \mathcal{H}_{\mathtt{D},\varepsilon}(\Omega)^{\perp_{\varepsilon}} & \longrightarrow & \operatorname{rot} \mathring{\mathsf{R}}(\Omega) \times \mathsf{L}^2(\Omega) \\ E & \longmapsto & (\operatorname{rot} E, \operatorname{div} \varepsilon E) \end{array}.$$

The analogous statement holds for $c_{\mathtt{m},\mathtt{n},\varepsilon}$ as well.

The compact embeddings (2.3)-(2.5) hold for more general bounded domains with weaker regularity of the boundary Γ, such as domains with cone property, restricted cone property or just p-cusp-property. See, e.g., [1, 2, 20, 21, 22, 23, 24, 26, 27, 28, 13]. Note that the Maxwell compactness properties and hence the Maxwell estimates hold for mixed boundary conditions as well, see [10, 7, 9]. The boundedness of the underlying domain Ω is crucial, since one has to work in weighted Sobolev spaces in unbounded, like exterior domains, see [11, 12, 13, 14, 15, 17, 16, 20, 24].

As always in the theory of Maxwell's equations, we need another crucial tool, the Helmholtz or Weyl decompositions of vector fields into irrotational and solenoidal vector fields. We have

$$\begin{aligned} \mathsf{L}^2_{\varepsilon}(\Omega) &= \nabla \mathring{\mathsf{H}}^1(\Omega) \oplus_{\varepsilon} \varepsilon^{-1}\mathsf{D}_0(\Omega) \\ &= \mathring{\mathsf{R}}_0(\Omega) \oplus_{\varepsilon} \varepsilon^{-1} \operatorname{rot} \mathsf{R}(\Omega) \\ &= \nabla \mathring{\mathsf{H}}^1(\Omega) \oplus_{\varepsilon} \mathcal{H}_{\mathtt{D},\varepsilon}(\Omega) \oplus_{\varepsilon} \varepsilon^{-1} \operatorname{rot} \mathsf{R}(\Omega), \\ \mathsf{L}^2_{\varepsilon}(\Omega) &= \nabla \mathsf{H}^1(\Omega) \oplus_{\varepsilon} \varepsilon^{-1}\mathring{\mathsf{D}}_0(\Omega) \\ &= \mathsf{R}_0(\Omega) \oplus_{\varepsilon} \varepsilon^{-1} \operatorname{rot} \mathring{\mathsf{R}}(\Omega) \\ &= \nabla \mathsf{H}^1(\Omega) \oplus_{\varepsilon} \mathcal{H}_{\mathtt{N},\varepsilon}(\Omega) \oplus_{\varepsilon} \varepsilon^{-1} \operatorname{rot} \mathring{\mathsf{R}}(\Omega), \end{aligned}$$

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[*] $d_{\mathtt{D}} := \dim \mathcal{H}_{\mathtt{D},\varepsilon}(\Omega)$ is finite and independent of ε. In particular, $d_{\mathtt{D}}$ depends just on the topology of Ω. More precisely, $d_{\mathtt{D}} = \beta_2$, the second Betti number of Ω. A similar result holds also for the Neumann fields, i.e., $d_{\mathtt{N}} := \dim \mathcal{H}_{\mathtt{N},\varepsilon}(\Omega) = \beta_1$.

where $\oplus_\varepsilon$ denotes the orthogonal sum with respect the latter scalar product, and note

$$\begin{aligned}
\nabla\mathring{\mathsf{H}}^1(\Omega) &= \mathring{\mathsf{R}}_0(\Omega)\cap\mathcal{H}_{\mathtt{D},\varepsilon}(\Omega)^{\perp_\varepsilon}, &\qquad \varepsilon^{-1}\operatorname{rot}\mathsf{R}(\Omega) &= \varepsilon^{-1}\mathsf{D}_0(\Omega)\cap\mathcal{H}_{\mathtt{D},\varepsilon}(\Omega)^{\perp_\varepsilon},\\
\nabla\mathsf{H}^1(\Omega) &= \mathsf{R}_0(\Omega)\cap\mathcal{H}_{\mathtt{N},\varepsilon}(\Omega)^{\perp_\varepsilon}, &\qquad \varepsilon^{-1}\operatorname{rot}\mathring{\mathsf{R}}(\Omega) &= \varepsilon^{-1}\mathring{\mathsf{D}}_0(\Omega)\cap\mathcal{H}_{\mathtt{N},\varepsilon}(\Omega)^{\perp_\varepsilon}.
\end{aligned}$$

Moreover, with

$$\begin{aligned}
\mathcal{R}(\Omega) &:= \mathsf{R}(\Omega)\cap\operatorname{rot}\mathring{\mathsf{R}}(\Omega) = \mathsf{R}(\Omega)\cap\mathring{\mathsf{D}}_0(\Omega)\cap\mathcal{H}_{\mathtt{N}}(\Omega)^{\perp},\\
\mathring{\mathcal{R}}(\Omega) &:= \mathring{\mathsf{R}}(\Omega)\cap\operatorname{rot}\mathsf{R}(\Omega) = \mathring{\mathsf{R}}(\Omega)\cap\mathsf{D}_0(\Omega)\cap\mathcal{H}_{\mathtt{D}}(\Omega)^{\perp}
\end{aligned}$$

we see

$$\operatorname{rot}\mathsf{R}(\Omega) = \operatorname{rot}\mathcal{R}(\Omega),\quad \operatorname{rot}\mathring{\mathsf{R}}(\Omega) = \operatorname{rot}\mathring{\mathcal{R}}(\Omega).$$

Note that all occurring spaces are closed subspaces of $\mathsf{L}^2(\Omega)$, which follows immediately by the estimates (2.6)-(2.9). More details about the Helmholtz decompositions can be found e.g. in [13].

If Ω is even convex[†] we have some simplifications due to the vanishing of Dirichlet and Neumann fields, i.e., $\mathcal{H}_{\mathtt{D},\varepsilon}(\Omega) = \mathcal{H}_{\mathtt{N},\varepsilon}(\Omega) = \{0\}$. Then (2.8) and (2.9) simplify to

$$\forall\, E\in\mathring{\mathsf{R}}(\Omega)\cap\varepsilon^{-1}\mathsf{D}(\Omega)\qquad |E|_{\Omega,\varepsilon}\le c_{\mathtt{m},\mathtt{t},\varepsilon}\big(|\operatorname{rot}E|_\Omega^2+|\operatorname{div}\varepsilon E|_\Omega^2\big)^{1/2},\tag{2.10}$$

$$\forall\, H\in\mathsf{R}(\Omega)\cap\varepsilon^{-1}\mathring{\mathsf{D}}(\Omega)\qquad |H|_{\Omega,\varepsilon}\le c_{\mathtt{m},\mathtt{n},\varepsilon}\big(|\operatorname{rot}H|_\Omega^2+|\operatorname{div}\varepsilon H|_\Omega^2\big)^{1/2}\tag{2.11}$$

and we have

$$\mathring{\mathsf{R}}_0(\Omega)=\nabla\mathring{\mathsf{H}}^1(\Omega),\quad \mathsf{R}_0(\Omega)=\nabla\mathsf{H}^1(\Omega),\quad \mathsf{D}_0(\Omega)=\operatorname{rot}\mathsf{R}(\Omega),\quad \mathring{\mathsf{D}}_0(\Omega)=\operatorname{rot}\mathring{\mathsf{R}}(\Omega)$$

as well as the simple Helmholtz decompositions

$$\mathsf{L}^2_\varepsilon(\Omega)=\nabla\mathring{\mathsf{H}}^1(\Omega)\oplus_\varepsilon\varepsilon^{-1}\operatorname{rot}\mathsf{R}(\Omega),\quad \mathsf{L}^2_\varepsilon(\Omega)=\nabla\mathsf{H}^1(\Omega)\oplus_\varepsilon\varepsilon^{-1}\operatorname{rot}\mathring{\mathsf{R}}(\Omega).\tag{2.12}$$

The aim of this paper is to give a computable estimate for the two Maxwell constants $c_{\mathtt{m},\mathtt{t},\varepsilon}$ and $c_{\mathtt{m},\mathtt{n},\varepsilon}$.

3 The Maxwell Estimates

First, we have an estimate for irrotational fields, which is well known.

Lemma 1 *For all* $E\in\nabla\mathring{\mathsf{H}}^1(\Omega)\cap\varepsilon^{-1}\mathsf{D}(\Omega)$ *and all* $H\in\nabla\mathsf{H}^1(\Omega)\cap\varepsilon^{-1}\mathring{\mathsf{D}}(\Omega)$

$$|E|_{\Omega,\varepsilon}\le\underline{\varepsilon}\,c_{\mathtt{p},\circ}|\operatorname{div}\varepsilon E|_\Omega,\quad |H|_{\Omega,\varepsilon}\le\underline{\varepsilon}\,c_{\mathtt{p}}|\operatorname{div}\varepsilon H|_\Omega.$$

[†] Note that convex domains are always Lipschitz, see e.g. [8].

Proof Pick a scalar potential $\varphi \in \mathring{\mathsf{H}}^1(\Omega)$ with $E = \nabla\varphi$. Then, by (2.6)

$$\begin{aligned}|E|^2_{\Omega,\varepsilon} &= \langle \varepsilon E, \nabla\varphi\rangle_\Omega = -\langle \operatorname{div} \varepsilon E, \varphi\rangle_\Omega \le |\operatorname{div} \varepsilon E|_\Omega |\varphi|_\Omega \le c_{\mathsf{p},\circ} |\operatorname{div} \varepsilon E|_\Omega |\nabla\varphi|_\Omega \\ &= c_{\mathsf{p},\circ} |\operatorname{div} \varepsilon E|_\Omega |E|_\Omega \le \underline{\varepsilon} c_{\mathsf{p},\circ} |\operatorname{div} \varepsilon E|_\Omega |E|_{\Omega,\varepsilon}.\end{aligned}$$

Let $\varphi \in \mathsf{H}^1(\Omega)$ with $H = \nabla\varphi$ and $\varphi \perp \mathbb{R}$. Since $\varepsilon H \in \mathring{\mathsf{D}}(\Omega)$ we obtain as before and by (2.7)

$$\begin{aligned}|H|^2_{\Omega,\varepsilon} &= \langle \varepsilon H, \nabla\varphi\rangle_\Omega = -\langle \operatorname{div} \varepsilon H, \varphi\rangle_\Omega \le |\operatorname{div} \varepsilon H|_\Omega |\varphi|_\Omega \le c_{\mathsf{p}} |\operatorname{div} \varepsilon H|_\Omega |\nabla\varphi|_\Omega \\ &= c_{\mathsf{p}} |\operatorname{div} \varepsilon H|_\Omega |H|_\Omega \le \underline{\varepsilon} c_{\mathsf{p}} |\operatorname{div} \varepsilon H|_\Omega |H|_{\Omega,\varepsilon},\end{aligned}$$

which finishes the proof. □

Remark 2 *Without any change, Lemma 1 extends to Lipschitz domains $\Omega \subset \mathbb{R}^N$ of arbitrary dimension.*

To get similar estimates for solenoidal vector fields we need a crucial lemma from [1, Theorem 2.17], see also [25, 8, 6, 4] for related partial results.

Lemma 3 *Let Ω be convex and $E \in \mathring{\mathsf{R}}(\Omega) \cap \mathsf{D}(\Omega)$ or $E \in \mathsf{R}(\Omega) \cap \mathring{\mathsf{D}}(\Omega)$. Then $E \in \mathsf{H}^1(\Omega)$ and*

$$|\nabla E|^2_\Omega \le |\operatorname{rot} E|^2_\Omega + |\operatorname{div} E|^2_\Omega. \tag{3.1}$$

We note that for $E \in \mathring{\mathsf{H}}^1(\Omega)$ it is clear that for any domain $\Omega \subset \mathbb{R}^3$

$$|\nabla E|^2_\Omega = |\operatorname{rot} E|^2_\Omega + |\operatorname{div} E|^2_\Omega \tag{3.2}$$

holds since $-\Delta = \operatorname{rot}\operatorname{rot} - \nabla \operatorname{div}$. This formula is no longer valid if E has just the tangential or normal boundary condition but for convex domains the inequality (3.1) remains true.

Lemma 4 *Let Ω be convex. For all vector fields $E \in \mathring{\mathsf{R}}(\Omega) \cap \varepsilon^{-1} \operatorname{rot} \mathsf{R}(\Omega)$ and all vector fields $H \in \mathsf{R}(\Omega) \cap \varepsilon^{-1} \operatorname{rot} \mathring{\mathsf{R}}(\Omega)$*

$$|E|_{\Omega,\varepsilon} \le \bar{\varepsilon} c_{\mathsf{p}} |\operatorname{rot} E|_\Omega, \quad |H|_{\Omega,\varepsilon} \le \bar{\varepsilon} c_{\mathsf{p}} |\operatorname{rot} H|_\Omega.$$

Proof Since $\varepsilon E \in \operatorname{rot} \mathsf{R}(\Omega) = \operatorname{rot} \mathcal{R}(\Omega)$ there exists a vector potential field $\Phi \in \mathcal{R}(\Omega)$ with $\operatorname{rot} \Phi = \varepsilon E$ and $\Phi \in \mathsf{H}^1(\Omega)$ by Lemma 3 since $\mathcal{R}(\Omega) = \mathsf{R}(\Omega) \cap \mathring{\mathsf{D}}_0(\Omega)$. Moreover, $\Phi = \operatorname{rot} \Psi$ can be represented by some $\Psi \in \mathring{\mathsf{R}}(\Omega)$. Hence, for any constant vector $a \in \mathbb{R}^3$ we have $\langle \Phi, a\rangle_\Omega = \langle \operatorname{rot} \Psi, a\rangle_\Omega = 0$. Thus, Φ belongs to $\mathsf{H}^1(\Omega) \cap (\mathbb{R}^3)^\perp$. Then, since $E \in \mathring{\mathsf{R}}(\Omega)$ and by Lemma 3 we get

$$\begin{aligned}|E|^2_{\Omega,\varepsilon} &= \langle E, \varepsilon E\rangle_\Omega = \langle E, \operatorname{rot} \Phi\rangle_\Omega = \langle \operatorname{rot} E, \Phi\rangle_\Omega \le |\operatorname{rot} E|_\Omega |\Phi|_\Omega \le c_{\mathsf{p}} |\operatorname{rot} E|_\Omega |\nabla\Phi|_\Omega \\ &\le c_{\mathsf{p}} |\operatorname{rot} E|_\Omega |\operatorname{rot} \Phi|_\Omega = c_{\mathsf{p}} |\operatorname{rot} E|_\Omega |\varepsilon E|_\Omega \le \bar{\varepsilon} c_{\mathsf{p}} |\operatorname{rot} E|_\Omega |E|_{\Omega,\varepsilon}.\end{aligned}$$

Since $\varepsilon H \in \operatorname{rot} \mathring{\mathsf{R}}(\Omega)$ there exists a vector potential $\Phi \in \mathring{\mathsf{R}}(\Omega)$ with $\operatorname{rot} \Phi = \varepsilon H$. Using the Helmholtz decomposition $\mathsf{L}^2(\Omega) = \mathsf{R}_0(\Omega) \oplus \operatorname{rot} \mathring{\mathsf{R}}(\Omega)$, we decompose

$$\mathsf{R}(\Omega) \ni H = H_0 + H_{\mathrm{rot}} \in \mathsf{R}_0(\Omega) \oplus \mathcal{R}(\Omega).$$

Then, $\operatorname{rot} H_{\mathrm{rot}} = \operatorname{rot} H$ and again by Lemma 3 we see $H_{\mathrm{rot}} \in \mathsf{H}^1(\Omega)$. Let $a \in \mathbb{R}^3$ such that $H_{\mathrm{rot}} - a \in \mathsf{H}^1(\Omega) \cap (\mathbb{R}^3)^{\perp}$. Since $\Phi \in \mathring{\mathsf{R}}(\Omega)$ and $\langle \operatorname{rot} \Phi, H_0 \rangle_\Omega = 0 = \langle \operatorname{rot} \Phi, a \rangle_\Omega$ as well as by Lemma 3 we obtain

$$\begin{aligned} |H|^2_{\Omega,\varepsilon} = \langle \varepsilon H, H \rangle_\Omega = \langle \operatorname{rot} \Phi, H \rangle_\Omega = \langle \operatorname{rot} \Phi, H_{\mathrm{rot}} - a \rangle_\Omega \le |\varepsilon H|_\Omega |H_{\mathrm{rot}} - a|_\Omega \\ \le c_{\mathtt{p}} |\varepsilon H|_\Omega |\nabla H_{\mathrm{rot}}|_\Omega \le \overline{\varepsilon} c_{\mathtt{p}} |H|_{\Omega,\varepsilon} |\operatorname{rot} H_{\mathrm{rot}}|_\Omega = \overline{\varepsilon} c_{\mathtt{p}} |H|_{\Omega,\varepsilon} |\operatorname{rot} H|_\Omega, \end{aligned}$$

completing the proof. □

Remark 5 *It is well known that Lemma 4 holds in two dimensions for any Lipschitz domain $\Omega \subset \mathbb{R}^2$. This follows immediately from Lemma 1 if we take into account that in two dimensions the rotation* rot *is given by the divergence* div *after 90°-rotation of the vector field to which it is applied. We refer to the appendix for details.*

Theorem 6 *Let Ω be convex. Then, for all vector fields $E \in \mathring{\mathsf{R}}(\Omega) \cap \varepsilon^{-1}\mathsf{D}(\Omega)$ and all vector fields $H \in \mathsf{R}(\Omega) \cap \varepsilon^{-1}\mathring{\mathsf{D}}(\Omega)$*

$$|E|^2_{\Omega,\varepsilon} \le \underline{\varepsilon}^2 c^2_{\mathtt{p},0} |\operatorname{div} \varepsilon E|^2_\Omega + \overline{\varepsilon}^2 c^2_{\mathtt{p}} |\operatorname{rot} E|^2_\Omega, \quad |H|^2_{\Omega,\varepsilon} \le \underline{\varepsilon}^2 c^2_{\mathtt{p}} |\operatorname{div} \varepsilon H|^2_\Omega + \overline{\varepsilon}^2 c^2_{\mathtt{p}} |\operatorname{rot} H|^2_\Omega.$$

Thus, $c_{\mathtt{m},\mathtt{t},\varepsilon} \le \max\{\underline{\varepsilon} c_{\mathtt{p},\circ}, \overline{\varepsilon} c_{\mathtt{p}}\}$ and

$$c_{\mathtt{m},\mathtt{t},\varepsilon}, c_{\mathtt{m},\mathtt{n},\varepsilon} \le \hat{\varepsilon} c_{\mathtt{p}} \le \hat{\varepsilon} \operatorname{diam}(\Omega)/\pi.$$

Proof By the Helmholtz decomposition (2.12) we have

$$\mathring{\mathsf{R}}(\Omega) \cap \varepsilon^{-1}\mathsf{D}(\Omega) \ni E = E_\nabla + E_{\mathrm{rot}} \in \nabla \mathring{\mathsf{H}}^1(\Omega) \oplus_\varepsilon \varepsilon^{-1} \operatorname{rot} \mathsf{R}(\Omega)$$

with $E_\nabla \in \nabla \mathring{\mathsf{H}}^1(\Omega) \cap \varepsilon^{-1}\mathsf{D}(\Omega)$ and $E_{\mathrm{rot}} \in \mathring{\mathsf{R}}(\Omega) \cap \varepsilon^{-1} \operatorname{rot} \mathsf{R}(\Omega)$ as well as

$$\operatorname{div} \varepsilon E_\nabla = \operatorname{div} \varepsilon E, \quad \operatorname{rot} E_{\mathrm{rot}} = \operatorname{rot} E.$$

By Lemma 1 and Lemma 4 and orthogonality we obtain

$$|E|^2_{\Omega,\varepsilon} = |E_\nabla|^2_{\Omega,\varepsilon} + |E_{\mathrm{rot}}|^2_{\Omega,\varepsilon} \le \underline{\varepsilon}^2 c^2_{\mathtt{p},\circ} |\operatorname{div} \varepsilon E|^2_\Omega + \overline{\varepsilon}^2 c^2_{\mathtt{p}} |\operatorname{rot} E|^2_\Omega.$$

Similarly we have

$$\mathsf{R}(\Omega) \cap \varepsilon^{-1}\mathring{\mathsf{D}}(\Omega) \ni H = H_\nabla + H_{\mathrm{rot}} \in \nabla \mathsf{H}^1(\Omega) \oplus_\varepsilon \varepsilon^{-1} \operatorname{rot} \mathring{\mathsf{R}}(\Omega)$$

with $H_\nabla \in \mathsf{H}^1(\Omega) \cap \varepsilon^{-1}\overset{\circ}{\mathsf{D}}(\Omega)$ and $H_{\mathrm{rot}} \in \mathsf{R}(\Omega) \cap \varepsilon^{-1} \operatorname{rot} \overset{\circ}{\mathsf{R}}(\Omega)$ as well as

$$\operatorname{div} \varepsilon H_\nabla = \operatorname{div} \varepsilon H, \quad \operatorname{rot} H_{\mathrm{rot}} = \operatorname{rot} H.$$

By Lemma 1 and Lemma 4

$$|H|_{\Omega,\varepsilon}^2 = |H_\nabla|_{\Omega,\varepsilon}^2 + |H_{\mathrm{rot}}|_{\Omega,\varepsilon}^2 \leq \underline{\varepsilon}^2 c_{\mathtt{p}}^2 |\operatorname{div} \varepsilon H|_\Omega^2 + \overline{\varepsilon}^2 c_{\mathtt{p}}^2 |\operatorname{rot} H|_\Omega^2,$$

which finishes the proof. □

In the case $\varepsilon = \mathrm{id}$ we can estimate the Maxwell constants also from below.

Theorem 7 *Let Ω be convex. If $\varepsilon = \mathrm{id}$ then*

$$c_{\mathtt{p},0} \leq c_{\mathtt{m},\mathtt{t}}, c_{\mathtt{m},\mathtt{n}} \leq c_{\mathtt{p}} \leq \operatorname{diam}(\Omega)/\pi.$$

Proof By Theorem 6 we just have to show the lower bound. Looking at (2.10) and (2.11) we get

$$\frac{1}{c_{\mathtt{m},\mathtt{t}}^2} = \inf_{0 \neq E \in \overset{\circ}{\mathsf{R}}(\Omega) \cap \mathsf{D}(\Omega)} \frac{|\operatorname{rot} E|_\Omega^2 + |\operatorname{div} E|_\Omega^2}{|E|_\Omega^2}, \quad \frac{1}{c_{\mathtt{m},\mathtt{n}}^2} = \inf_{0 \neq E \in \mathsf{R}(\Omega) \cap \overset{\circ}{\mathsf{D}}(\Omega)} \frac{|\operatorname{rot} E|_\Omega^2 + |\operatorname{div} E|_\Omega^2}{|E|_\Omega^2}.$$

Thus

$$\frac{1}{c_{\mathtt{m},\mathtt{t}}^2}, \frac{1}{c_{\mathtt{m},\mathtt{n}}^2} \leq \inf_{0 \neq E \in \overset{\circ}{\mathsf{H}}{}^1(\Omega)} \frac{|\operatorname{rot} E|_\Omega^2 + |\operatorname{div} E|_\Omega^2}{|E|_\Omega^2} = \inf_{0 \neq E \in \overset{\circ}{\mathsf{H}}{}^1(\Omega)} \frac{|\nabla E|_\Omega^2}{|E|_\Omega^2} = \frac{1}{c_{\mathtt{p},0}^2},$$

completing the proof. □

Remark 8

(i) *It is unclear but most probable that $c_{\mathtt{p},0} < c_{\mathtt{m},\mathtt{t}} < c_{\mathtt{m},\mathtt{n}} = c_{\mathtt{p}}$ holds. In a forthcoming publication [18] we will show more and sharper estimates on the Maxwell constants, showing additional and sharp relations between the Maxwell and the Poincaré/Friedrichs/Steklov constants.*

(ii) *It is well known that*

$$c_{\mathtt{p},\circ} = 1/\sqrt{\lambda_1}, \quad c_{\mathtt{p}} = 1/\sqrt{\mu_2}$$

hold, where λ_1 is the first Dirichlet and μ_2 the second Neumann eigenvalue of the Laplacian. We note that by Theorem 7 we have given a new proof of the estimate

$$0 < \mu_2 \leq \lambda_1,$$

which is of course well known, since even $0 < \mu_{n+1} < \lambda_n$, $n \geq 1$, holds, see, e.g., [5] and the literature cited there. Moreover, by Theorem 7 the eigenvalues of the different Maxwell operators lie between $\sqrt{\mu_2}$ and $\sqrt{\lambda_1}$.

(iii) *Our results extend also to all polyhedrons which allow the* $\mathsf{H}^1(\Omega)$*-regularity of the Maxwell spaces* $\overset{\circ}{\mathsf{R}}(\Omega) \cap \mathsf{D}(\Omega)$ *and* $\mathsf{R}(\Omega) \cap \overset{\circ}{\mathsf{D}}(\Omega)$ *or to domains whose boundaries consist of combinations of convex boundary parts and polygonal parts which allow the* $\mathsf{H}^1(\Omega)$*-regularity. Is is shown in [4, Theorem 4.1] that* (3.1)*, even* (3.2)*, still holds for all* $E \in \mathsf{H}^1(\Omega) \cap \overset{\circ}{\mathsf{R}}(\Omega)$ *or* $E \in \mathsf{H}^1(\Omega) \cap \overset{\circ}{\mathsf{D}}(\Omega)$ *if* Ω *is a polyhedron*[‡]*. We note that even some non-convex polyhedrons admit the* $\mathsf{H}^1(\Omega)$*-regularity of the Maxwell spaces depending on the angle of the corners, which are not allowed to by too pointy.*

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A Appendix: The Maxwell Estimates in Two Dimensions

Finally, we want to note that similar but simpler results hold in two dimensions as well. More precisely, for $N = 2$ the Maxwell constants can be estimated by the Poincaré constants in any bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$. Although this is quite well known, we present the results for convenience and completeness.

As noted before, Lemma 1 holds in any dimension. In two dimensions the rotation rot differs from the divergence div just by a 90°-rotation R given by

$$R := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad R^2 = -\operatorname{id}, \quad R^\top = -R = R^{-1}.$$

The same holds for the co-gradient $\lhd := \operatorname{rot}^*$ (as formal adjoint) and the gradient ∇. More precisely, for smooth functions u and smooth vector fields v we have

$$\begin{aligned} \operatorname{rot} v &= \operatorname{div} Rv = \partial_1 v_2 - \partial_2 v_1, & \lhd u &= R\nabla u = \begin{bmatrix} \partial_2 u \\ -\partial_1 u \end{bmatrix}, \\ \operatorname{div} v &= -\operatorname{rot} Rv, & \nabla u &= -R \lhd u \end{aligned}$$

and thus also $-\Delta u = -\operatorname{div} \nabla u = \operatorname{div} RR\nabla u = \operatorname{rot} \lhd u$. For the vector Laplacian we have $-\Delta v = \lhd \operatorname{rot} - \nabla \operatorname{div}$. Furthermore,

$$v \in \mathsf{R}(\Omega) \quad \Leftrightarrow \quad Rv \in \mathsf{D}(\Omega), \qquad v \in \mathring{\mathsf{R}}(\Omega) \quad \Leftrightarrow \quad Rv \in \mathring{\mathsf{D}}(\Omega).$$

The Helmholtz decompositions read

$$\begin{aligned} \mathsf{L}^2(\Omega) &= \nabla \mathring{\mathsf{H}}^1(\Omega) \oplus_\varepsilon \varepsilon^{-1} \mathsf{D}_0(\Omega) \\ &= \mathring{\mathsf{R}}_0(\Omega) \oplus_\varepsilon \varepsilon^{-1} \lhd \mathsf{H}^1(\Omega) \\ &= \nabla \mathring{\mathsf{H}}^1(\Omega) \oplus_\varepsilon \mathcal{H}_{\mathtt{D},\varepsilon}(\Omega) \oplus_\varepsilon \varepsilon^{-1} \lhd \mathsf{H}^1(\Omega), \\ \mathsf{L}^2(\Omega) &= \nabla \mathsf{H}^1(\Omega) \oplus_\varepsilon \varepsilon^{-1} \mathring{\mathsf{D}}_0(\Omega) \\ &= \mathsf{R}_0(\Omega) \oplus_\varepsilon \varepsilon^{-1} \lhd \mathring{\mathsf{H}}^1(\Omega) \\ &= \nabla \mathsf{H}^1(\Omega) \oplus_\varepsilon \mathcal{H}_{\mathtt{N},\varepsilon}(\Omega) \oplus_\varepsilon \varepsilon^{-1} \lhd \mathring{\mathsf{H}}^1(\Omega) \end{aligned}$$

and we note

$$\begin{aligned}
\nabla\mathring{\mathsf{H}}^1(\Omega)&=\mathring{\mathsf{R}}_0(\Omega)\cap\mathcal{H}_{\mathtt{D},\varepsilon}(\Omega)^{\perp_\varepsilon}, &\qquad \varepsilon^{-1}\lhd\mathsf{H}^1(\Omega)&=\varepsilon^{-1}\mathsf{D}_0(\Omega)\cap\mathcal{H}_{\mathtt{D},\varepsilon}(\Omega)^{\perp_\varepsilon},\\
\nabla\mathsf{H}^1(\Omega)&=\mathsf{R}_0(\Omega)\cap\mathcal{H}_{\mathtt{N},\varepsilon}(\Omega)^{\perp_\varepsilon}, &\qquad \varepsilon^{-1}\lhd\mathring{\mathsf{H}}^1(\Omega)&=\varepsilon^{-1}\mathring{\mathsf{D}}_0(\Omega)\cap\mathcal{H}_{\mathtt{N},\varepsilon}(\Omega)^{\perp_\varepsilon}.
\end{aligned}$$

We also need the matrix $\varepsilon_R:=-R\varepsilon R$, which fulfills the same estimates as ε, i.e., for all $E\in\mathsf{L}^2(\Omega)$

$$\underline{\varepsilon}^{-2}|E|_\Omega^2\leq\langle\varepsilon_R E,E\rangle_\Omega\leq\overline{\varepsilon}^2|E|_\Omega^2,$$

since $\langle\varepsilon_R E,E\rangle_\Omega=\langle\varepsilon RE,RE\rangle_\Omega$ and $|RE|_\Omega=|E|_\Omega$. But then the inverse ε_R^{-1} satisfies for all $E\in\mathsf{L}^2(\Omega)$

$$\overline{\varepsilon}^{-2}|E|_\Omega^2\leq\langle\varepsilon_R^{-1}E,E\rangle_\Omega\leq\underline{\varepsilon}^2|E|_\Omega^2,$$

which immediately follows by (2.2), i.e.,

$$\langle\varepsilon_R^{-1}E,E\rangle_\Omega=|\varepsilon_R^{-1/2}E|_\Omega^2\left\{\begin{array}{l}\leq\underline{\varepsilon}^2\langle\varepsilon_R\varepsilon_R^{-1/2}E,\varepsilon_R^{-1/2}E\rangle_\Omega=\underline{\varepsilon}^2|E|_\Omega^2\\ \geq\overline{\varepsilon}^{-2}\langle\varepsilon_R\varepsilon_R^{-1/2}E,\varepsilon_R^{-1/2}E\rangle_\Omega=\overline{\varepsilon}^{-2}|E|_\Omega^2\end{array}\right..$$

Hence, for the inverse matrix $\varepsilon_R^{-1}=-R\varepsilon^{-1}R$ simply $\underline{\varepsilon}$ and $\overline{\varepsilon}$ has to be exchanged. Furthermore, we have $\varepsilon_R^{\pm1/2}=-R\varepsilon^{\pm1/2}R$.

For the solenoidal fields we have the following:

Lemma 9 *For all* $E\in\mathring{\mathsf{R}}(\Omega)\cap\varepsilon^{-1}\lhd\mathsf{H}^1(\Omega)$ *and all* $H\in\mathsf{R}(\Omega)\cap\varepsilon^{-1}\lhd\mathring{\mathsf{H}}^1(\Omega)$

$$|E|_{\Omega,\varepsilon}\leq\overline{\varepsilon}c_{\mathtt{p}}|\operatorname{rot}E|_\Omega,\qquad|H|_{\Omega,\varepsilon}\leq\overline{\varepsilon}c_{\mathtt{p},\circ}|\operatorname{rot}H|_\Omega.$$

Proof Since $RE\in\mathring{\mathsf{D}}(\Omega)$ and $R\varepsilon E\in\nabla\mathsf{H}^1(\Omega)$ we have $R\varepsilon E\in\nabla\mathsf{H}^1(\Omega)\cap\varepsilon_R\mathring{\mathsf{D}}(\Omega)$. By Lemma 1 (interchanging $\underline{\varepsilon}$ and $\overline{\varepsilon}$) we get

$$\begin{aligned}
|E|_{\Omega,\varepsilon}&=|\varepsilon^{1/2}E|_\Omega=|R\varepsilon^{-1/2}\varepsilon E|_\Omega=|\varepsilon_R^{-1/2}R\varepsilon E|_\Omega=|R\varepsilon E|_{\Omega,\varepsilon_R^{-1}}\\
&\leq\overline{\varepsilon}c_{\mathtt{p}}|\operatorname{div}\varepsilon_R^{-1}R\varepsilon E|_\Omega=\overline{\varepsilon}c_{\mathtt{p}}|\operatorname{rot}E|_\Omega.
\end{aligned}$$

Analogously, as $RH\in\mathsf{D}(\Omega)$ and $R\varepsilon H\in\nabla\mathring{\mathsf{H}}^1(\Omega)$ we have $R\varepsilon H\in\nabla\mathring{\mathsf{H}}^1(\Omega)\cap\varepsilon_R\mathsf{D}(\Omega)$. Again by Lemma 1 (and again interchanging $\underline{\varepsilon}$ and $\overline{\varepsilon}$) we get

$$\begin{aligned}
|H|_{\Omega,\varepsilon}&=|\varepsilon^{1/2}H|_\Omega=|R\varepsilon^{-1/2}\varepsilon H|_\Omega=|\varepsilon_R^{-1/2}R\varepsilon H|_\Omega=|R\varepsilon H|_{\Omega,\varepsilon_R^{-1}}\\
&\leq\overline{\varepsilon}c_{\mathtt{p},\circ}|\operatorname{div}\varepsilon_R^{-1}R\varepsilon H|_\Omega=\overline{\varepsilon}c_{\mathtt{p},\circ}|\operatorname{rot}H|_\Omega,
\end{aligned}$$

which completes the proof. □

Finally, the main result in proved as Theorem 6 and 7, but taking into account that there are now possibly Dirichlet and Neumann fields.

Theorem 10 *For all* $E \in \mathring{\mathsf{R}}(\Omega) \cap \varepsilon^{-1}\mathsf{D}(\Omega)$ *and all* $H \in \mathsf{R}(\Omega) \cap \varepsilon^{-1}\mathring{\mathsf{D}}(\Omega)$

$$|E - \pi_{\mathtt{D}}E|^2_{\Omega,\varepsilon} \leq \underline{\varepsilon}^2 c^2_{\mathtt{p},\circ} |\operatorname{div} \varepsilon E|^2_{\Omega} + \overline{\varepsilon}^2 c^2_{\mathtt{p}} |\operatorname{rot} E|^2_{\Omega},$$
$$|H - \pi_{\mathtt{N}}H|^2_{\Omega,\varepsilon} \leq \underline{\varepsilon}^2 c^2_{\mathtt{p}} |\operatorname{div} \varepsilon H|^2_{\Omega} + \overline{\varepsilon}^2 c^2_{\mathtt{p},\circ} |\operatorname{rot} H|^2_{\Omega}.$$

Thus, $c_{\mathtt{m},\mathtt{t},\varepsilon} \leq \max\{\underline{\varepsilon} c_{\mathtt{p},\circ}, \overline{\varepsilon} c_{\mathtt{p}}\}$, $c_{\mathtt{m},\mathtt{n},\varepsilon} \leq \max\{\underline{\varepsilon} c_{\mathtt{p}}, \overline{\varepsilon} c_{\mathtt{p},\circ}\}$ *and*

$$c_{\mathtt{m},\mathtt{t},\varepsilon}, c_{\mathtt{m},\mathtt{n},\varepsilon} \leq \hat{\varepsilon} c_{\mathtt{p}}.$$

If Ω *is simply connected and* $\varepsilon = \mathrm{id}$ *then*

$$c_{\mathtt{p},0} \leq c_{\mathtt{m},\mathtt{t}}, c_{\mathtt{m},\mathtt{n}} \leq c_{\mathtt{p}} \leq \operatorname{diam}(\Omega)/\pi.$$

Proof Using the Helmholtz decomposition we have

$$\mathring{\mathsf{R}}(\Omega) \cap \varepsilon^{-1}\mathsf{D}(\Omega) \cap \mathcal{H}_{\mathtt{D},\varepsilon}(\Omega)^{\perp_\varepsilon} \ni E - \pi_{\mathtt{D}}E = E_{\nabla} + E_{\lhd} \in \nabla\mathring{\mathsf{H}}^1(\Omega) \oplus_\varepsilon \varepsilon^{-1} \lhd \mathsf{H}^1(\Omega)$$

with $E_{\nabla} \in \nabla\mathring{\mathsf{H}}^1(\Omega) \cap \varepsilon^{-1}\mathsf{D}(\Omega)$ and $E_{\lhd} \in \mathring{\mathsf{R}}(\Omega) \cap \varepsilon^{-1} \lhd \mathsf{H}^1(\Omega)$ as well as

$$\operatorname{div} \varepsilon E_{\nabla} = \operatorname{div} \varepsilon E, \quad \operatorname{rot} E_{\lhd} = \operatorname{rot} E.$$

Thus, by Lemma 1 and Lemma 9 as well as orthogonality we obtain

$$|E - \pi_{\mathtt{D}}E|^2_{\Omega,\varepsilon} = |E_{\nabla}|^2_{\Omega,\varepsilon} + |E_{\lhd}|^2_{\Omega,\varepsilon} \leq \underline{\varepsilon}^2 c^2_{\mathtt{p},\circ} |\operatorname{div} \varepsilon E|^2_{\Omega} + \overline{\varepsilon}^2 c^2_{\mathtt{p}} |\operatorname{rot} E|^2_{\Omega}.$$

Analogously, we decompose

$$\mathsf{R}(\Omega) \cap \varepsilon^{-1}\mathring{\mathsf{D}}(\Omega) \cap \mathcal{H}_{\mathtt{N},\varepsilon}(\Omega)^{\perp_\varepsilon} \ni H - \pi_{\mathtt{N}}H = H_{\nabla} + H_{\lhd} \in \nabla\mathsf{H}^1(\Omega) \oplus_\varepsilon \varepsilon^{-1} \lhd \mathring{\mathsf{H}}^1(\Omega)$$

with $H_{\nabla} \in \nabla\mathsf{H}^1(\Omega) \cap \varepsilon^{-1}\mathring{\mathsf{D}}(\Omega)$ and $H_{\lhd} \in \mathsf{R}(\Omega) \cap \varepsilon^{-1} \lhd \mathring{\mathsf{H}}^1(\Omega)$ as well as

$$\operatorname{div} \varepsilon H_{\nabla} = \operatorname{div} \varepsilon H, \quad \operatorname{rot} H_{\lhd} = \operatorname{rot} H.$$

As before, by Lemma 1, Lemma 9 and orthogonality we see

$$|H - \pi_{\mathtt{N}}H|^2_{\Omega,\varepsilon} = |H_{\nabla}|^2_{\Omega,\varepsilon} + |H_{\lhd}|^2_{\Omega,\varepsilon} \leq \underline{\varepsilon}^2 c^2_{\mathtt{p}} |\operatorname{div} \varepsilon H|^2_{\Omega} + \overline{\varepsilon}^2 c^2_{\mathtt{p},\circ} |\operatorname{rot} H|_{\Omega},$$

yielding the assertion for the upper bounds. For the lower bounds, we assume that Ω is simply connected, which is in two dimensions equivalent to a connected boundary. Hence, there are no Dirichlet or Neumann fields. If moreover $\varepsilon = \mathrm{id}$, we can use the same arguments as in the proof of Theorem 7, which completes the proof. □

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