Recursive Constructions for 3-Designs and Resolvable 3-Designs

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Dedicated to S. S. Shrikhande.

Abstract

Inspired by the doubling construction method for Steiner quadruple systems and also by a construction of Driessen for 3-designs, we present several recursive constructions for 3-designs and resolvable 3-designs. The construction methods assume the existence of resolvable 3-designs and certain appropriate other 3-designs. They prove to be very useful, as we can construct a large number of new infinite families of 3-designs. Among others we prove, for instance, that for any integer $n \geq 3$, there is a family $\mathcal{F}_n$ of resolvable 3-designs having parameters $3-(2^j.3.2^n, 2^n, (2^{n-1}-1)(2^n-1)\prod_{i=2}^{n-1}(2^{j-i}.3.2^n-1))$, for all $j \geq 0$. A list of parameters for newly constructed 3-designs is included.

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1 Introduction

All the designs considered in this paper are simple, i.e., no repeated blocks are allowed. A resolvable $t-(v, k, \lambda)$ design D here means that the blocks of D can be partitioned into parallel classes, each class consists of v/k pairwise disjoint blocks. For notation and definitions of $t-$designs we refer to [5]. Our aim is to present recursive methods for constructing 3-designs and resolvable 3-designs. Our constructions are inspired by the doubling construction for Steiner quadruple systems, which goes as far back as Witt (1938) [9] and a construction of Driessen [4] for 3-designs which can be considered as a generalization of the doubling construction. The paper is organized as follows. Construction I in section 2 is a general form of the doubling construction for 3-designs. The method turns out to be useful as many new families of 3-designs, which are presented in subsection 2.1, are constructed using this procedure. Constructions of resolvable 3-designs are shown in subsection 2.2, wherein applications of Construction I and further methods are explored, and many new families of resolvable 3-designs are displayed. Construction II in section 3 and Construction III in section 4 are methods which provide 3-designs whose number of points is not necessarily divisible by the block size.

In section 5 we show three special constructions for 3-designs with block sizes 5, 7 and 8. The paper is closed with an Appendix containing a list of parameters for newly constructed 3-designs.

2 Construction I

The construction in this section is a most natural generalization of the doubling construction for Steiner quadruple systems.

Let $D = (X, \mathcal{B})$ be a resolvable $3-(v,k,\lambda)$ (resp. 2-(v,2,1)) design, for $k \geq 3$ (resp. $k = 2$).

Let $\pi_1, \ldots, \pi_r$ denote the r parallel classes of D. Define a distance between any two parallel classes π_i and π_j by $d(\pi_i, \pi_j) = \min\{|i-j|, r-|i-j|\}$.

Let $\tilde{D} = (\tilde{X}, \tilde{\mathcal{B}})$ be a copy of D such that $X \cap \tilde{X} = \emptyset$. Let $D^* = (X, \mathcal{B}^*)$ be a $3-(v, 2k, \Lambda)$ design.

Define blocks on the point set $X \cup \tilde{X}$ as follows:

I. blocks of a copy of D^* defined on X;

II. blocks of a copy of D^* defined on $\tilde{X}$;

III. $B \cup \tilde{B}$ for any pair $B \in \pi_i$ and $\tilde{B} \in \tilde{\pi_j}$ with $\epsilon \leq d(\pi_i, \pi_j) \leq s$, $\epsilon = 0, 1$.

Case a: $k \geq 3$.

Any 3 points $a, b, c \in X$ (resp. $\tilde{a}, \tilde{b}, \tilde{c} \in \tilde{X}$) are contained in Λ blocks of type I (resp. type II) and in $(2s+1-\epsilon)\lambda\frac{v}{k}$ of type III.

Any 3 points $a, b, \tilde{c}$, where $a, b \in X$ and $\tilde{c} \in \tilde{X}$, (resp. $\tilde{a}, \tilde{b}, c$) are contained in $(2s+1-\epsilon)\lambda\frac{v-2}{k-2}$ blocks of type III.

The defined blocks form a 3-design if and only if $\Lambda + (2s+1-\epsilon)\lambda\frac{v}{k} = (2s+1-\epsilon)\lambda\frac{v-2}{k-2}$ or equivalently $(2s+1-\epsilon) = \frac{\Lambda k(k-2)}{2\lambda(v-k)}$. In this case we obtain a $3-(2v, 2k, \frac{\Lambda k(v-2)}{2(v-k)})$ design.

Case b: $k = 2$.

Here D is the trivial $2-(2m, 2, 1)$ design, and D^* is a $3-(2m, 4, \Lambda)$ design.

Any 3 points $a, b, c \in X$ (resp. $\tilde{a}, \tilde{b}, \tilde{c} \in \tilde{X}$) are contained in Λ blocks of type I.

Any 3 points $a, b, \tilde{c}$, where $a, b \in X$ and $\tilde{c} \in \tilde{X}$, (resp. $\tilde{a}, \tilde{b}, c$) are contained in $(2s+1-\epsilon)$ blocks of type III.

The condition for which the defined blocks form a 3-design is $\Lambda = (2s+1-\epsilon) \leq 2m-1$, and the constructed design has parameters $3-(4m, 4, \Lambda)$.

It is clear from the construction that the resulting design is resolvable if D^* is resolvable.

We summarize the construction in the following theorem.

Theorem 2.1 *(i) If there exists a $3-(2m, 4, \Lambda)$ design D^* with $\Lambda \leq 2m-1$, then there exists a $3-(4m, 4, \Lambda)$ design C.*

(ii) Suppose there exists a resolvable $3-(v,k,\lambda)$ design D and a $3-(v, 2k, \Lambda)$ design D^* such that $\frac{\Lambda k(k-2)}{2\lambda(v-k)}$ is an integer $\leq r$, where r is the number of parallel classes of D, then there exists a $3-(2v, 2k, \Theta)$ design C with $\Theta = \frac{\Lambda k(v-2)}{2(v-k)}$.

Moreover, if D^* is resolvable, then C is resolvable for both cases (i) and (ii).

Remark 2.1 If D^* is chosen to be a $3-(2m, 4, 1)$ design in Theorem 2.1 (i), then we have the doubling construction for Steiner quadruple systems.

If D^* is the trivial design in Theorem 2.1 and $\epsilon = 0$, then we have the construction of Driessen. It should be noted that the Driessen construction provides at most one 3-design from a given resolvable $3-(v,k,\lambda)$ design, whereas Construction I may yield a large number of 3-designs from a given one. As an example, take the trivial 3-(12,3,1) design for D and a $3-(12,6,m2)$ design for D^*, where $m \in \{1,2,\ldots,42\}$. The numerical condition of Theorem 2.1 is satisfied if $3|m$. Thus the resulting design C with parameters $3-(24,6,10m/3)$ is obtained for $m = 3,6,9,12,15,18,21,24,27,30,33,36,39,42$. The last value $m = 42$ corresponds to the design in Driessen construction.

2.1 Applications of Construction I

As a first example, take the 3-(15,3,1) design for D and a $3-(15,6,m20)$ design for D^*, where $m \in \{1,2,\ldots,11\}$. The condition that $\frac{\Lambda k(k-2)}{2\lambda(v-k)}$ is an integer implies that m is even. Hence the parameters of the resulting designs C are 3-(30,6,65), 3-(30,6,130), 3-(30,6,195), 3-(30,6,260) and 3-(30,6,325). These designs are indicated as unknown in the Handbook of Combinatorial Designs [5], p.57.

Thus we have

Theorem 2.2 *There is a* $3-(30,6,m5)$ *design for* $m = 13,26,39,52,65$.

In the same vein as Theorem 2.2 we can prove that $3-(32,8,m7)$ designs exist for $m = 1,\ldots,35$ by taking D as a resolvable 3-(16,4,1) design and D^* as a $3-(16,8,m3)$ design with $m = 1,\ldots,35$. Similarly, when D is a resolvable 3-(20,4,1) design and D^* is a $3-(20,8,m14)$ design, where $m = 1,\ldots,16$, the design C of parameters $3-(40,8,n63)$ can be constructed for all $m = 2n$ with $n = 1,\ldots,8$.

Hence we have the following results.

Theorem 2.3 *(i) There exists a* $3-(32,8,m7)$ *design for* $m = 1,\ldots,35$.

(ii) There exists a $3-(40,8,n63)$ *design for* $n = 1,\ldots,8$.

As another example, take the trivial $3-(2^n+1,3,1)$ design for D, where n is odd. D is resolvable after a theorem of Baranyai [2]. Take D^* as a $3-(2^n+1,6,10(2^n-2)/3)$ design with odd $n \geq 5$. D^* is obtained from a $4-(2^n+1,6,10)$ design constructed by Bierbrauer [3]. It is easy to check that $\frac{\Lambda k(k-2)}{2\lambda(v-k)} = (2s+1-\epsilon) = 5$ (i.e. $\epsilon = 0$). Theorem 2.1 then yields a $3-(2^{n+1}+2,6,5(2^n-1))$ design. Thus we have the following result.

Theorem 2.4 *There exists a* $3-(2^{n+1}+2,6,5(2^n-1))$ *design for all odd* $n \geq 5$.

We observe that the construction in Theorem 2.1 can produce infinite families of 3-designs when using it recursively.

As examples we illustrate the construction of two families of 3-designs with $k = 8$.

1. Let D_i be a resolvable $3-(2^i 20,4,1)$ design for $i \geq 0$. D_i is known to exist for all i, see [5] I.4.32. Let D_0^* be a $3-(20,8,28)$ design. Construction I with the pair (D_0, D_0^*) yields a $3-(40,8,63)$ design D_1^*. Applying Construction I for the pair (D_1, D_1^*) yields a $3-(80,8,133)$ design D_2^*. Repeat Construction I with the pair (D_2, D_2^*) and so on will provide a family of 3-designs having parameters $3-(2^i 20,8,7(2^{i-2}20-1))$ for all integers $i \geq 0$. To see this, we need to verify the divisibility condition for $\frac{\Lambda^{(i)}k(k-2)}{2\lambda^{(i)}(v_i-k)}$ and to compute $\Lambda^{(i)}$. Since $v_i = 2^i 20$, $\lambda^{(i)} = 1$ and $\Lambda^{(i)} = \frac{\Lambda^{(i-1)}k(v_{i-1}-2)}{2(v_{i-1}-k)}$, we have $\frac{\Lambda^{(i)}k(k-2)}{2\lambda^{(i)}(v_i-k)} = \frac{4\Lambda^{(i)}}{(v_i-4)} = \frac{4\Lambda^{(i-1)}}{(v_{i-1}-4)} = \ldots = \frac{4\Lambda^{(0)}}{(v_0-4)} = 7$. Hence, $\Lambda^{(i)} = 7(v_i-4)/4 = 7(2^{i-2}20-1)$ as desired.

2. In the same way, we will obtain a $3-(2^i28,8,7(2^{i-2}28-1))$ design for all $i \geq 0$ when starting with a resolvable $3-(28,4,1)$ design as D_0 and a $3-(28,8,42)$ design as D_0^*. Here we have $\Lambda^{(i)} = \frac{\Lambda^{(i-1)}2(v_{i-1}-2)}{(v_{i-1}-4)}$ and $\frac{\Lambda^{(i)}k(k-2)}{2\lambda^{(i)}(v_i-k)} = \frac{4\Lambda^{(i)}}{(v_i-4)} = \frac{4\Lambda^{(i-1)}}{(v_{i-1}-4)} = \cdots = \frac{4\Lambda^{(0)}}{(v_0-4)} = 7$.

Thus we have proved the following result.

Theorem 2.5 *For all $i \geq 0$ designs with the following parameters exist*

1. $3-(2^i20,8,7(2^{i-2}20-1))$,
2. $3-(2^i28,8,7(2^{i-2}28-1))$.

2.2 Constructions of resolvable 3-designs

In this section we investigate constructions for resolvable 3-designs. As shown in Theorem 2.1 if both D and D^* are resolvable, then so is the resulting design C. Whereas, if D^* is not resolvable, then, in general, C is not either. In the following, however, we prove that if $v = 3k$ and D^*, which is never resolvable in this case, is chosen in a particular way, then the resulting design C is resolvable. This result turns out to be very useful as it can be combined with Construction I to produce a great quantity of new families of resolvable 3-designs.

At first consider two simple but useful results related to resolvable t-designs with $v = 2k$.

Theorem 2.6 *If there is a resolvable $t-(2k,k,\lambda)$ design, then there is a resolvable $t-(2k,k,\binom{2k-t}{k-t}-\lambda)$ design.*

Proof. Let D be a resolvable $t-(2k,k,\lambda)$ design, then the supplementary design $\bar{D}$ consisting of all k-subsets not being a block of D is a $3-(2k,k,\binom{2k-t}{k-t}-\lambda)$ design. $\bar{D}$ is resolvable, because if $\bar{C}$ is a block of $\bar{D}$, then the complement $\bar{C}^*$ is also a block of $\bar{D}$, since otherwise $\bar{C}^*$, and therefore $\bar{C}$, would be both blocks of D, which is impossible. □

The next theorem about resolvable t-designs with $v = 2k$ is derived from a construction of Alltop [1].

Theorem 2.7 *If there exists a $(2t+1)-(2k,k,\lambda)$ design, then there exists a resolvable $(2t+1)-(2k,k,\lambda)$ design.*

Proof. Suppose that there is a $(2t+1)-(2k,k,\lambda)$ design $D = (X,\mathcal{B})$. If D is resolvable, then the theorem is proved. If not, let $D_z = (X_z,\mathcal{B}_1)$, $X_z = X - \{z\}$, be the derived design $2t-(2k-1,k-1,\lambda)$ of D at a point $z \in X$. Then $D^* = (X_z \cup \{z\}, \mathcal{B}_1^+ \cup \mathcal{B}_1^*)$ is a resolvable $(2t+1)-(2k,k,\lambda)$ design, where $\mathcal{B}_1^+ = \{B \cup \{z\},\ B \in \mathcal{B}_1\}$ and $\mathcal{B}_1^* = \{X - B,\ B \in \mathcal{B}_1\}$. □

As an illustration of Theorem 2.6 and Theorem 2.7, we present several small parameters of resolvable $3-(2k,k,\lambda)$ designs for $k \leq 10$ using known 3-designs given in [5].

Theorem 2.8 *There is a resolvable 3-design for the following parameters.*

(i) $3-(8,4,n)$, $n = 1,\ldots,5$;

(ii) $3-(10,5,n3)$, $n = 1,\ldots,7$;

(iii) $3-(12,6,n2)$, $n = 1,\ldots,42$;

(iv) $3-(14,7,n5)$, $n=1,\ldots,66$;

(v) $3-(16,8,n3)$, $n=1,\ldots,429$;

(vi) $3-(18,9,n7)$, $n=1,\ldots,715$;

(vii) $3-(20,10,n4)$, $n=1,\ldots,4862$.

As first examples for resolvable 3-designs obtained from Construction I we have

Theorem 2.9 *There is a resolvable 3-design for the following parameters:*

(i) $3-(24,6,n10)$, $n=1,\ldots,14$;

(ii) $3-(32,8,m7)$, $m=1,\ldots,35$.

Proof. (i) Take the trivial design 3-(12,3,1) for D and a resolvable $3-(12,6,m2)$ design for D^*, where $m=1,\ldots,42$. It is easily checked that if $3|m$, then the resulting design C has parameters $3-(24,6,\frac{m}{3}10)$.

(ii) In this case, D is a resolvable Steiner quadruple system 3-(16,4,1) and D^* is a resolvable $3-(16,8,m3)$, $m=1,\ldots,35$. □

We now consider the case $v=3k$ of Construction I.

Suppose there is a resolvable $3-(3k,k,\lambda)$ design D, $k\geq 3$. Take the complementary design of D for D^*. So, D^* is a $3-(3k,2k,\Lambda)$ design with $\Lambda=\lambda\binom{2k}{3}/\binom{k}{3}$. Note that D^* is never resolvable. It is now easy to verify that $\frac{\Lambda k(k-2)}{2\lambda(v-k)}=2k-1$, hence Construction I yields a design C with parameters $3-(6k,2k,\Theta)$, where $\Theta=\lambda(2k-1)(3k-2)/(k-2)$.

We show that C is resolvable. Let $\tilde{D}^*$ be a copy of D^* defined on $\tilde{X}$. First of all, note that D and D^* have the same number of blocks. Since $2s+1-\epsilon=2k-1$, we have $\epsilon=0$.

Type I

Let A_{i_1},A_{i_2},A_{i_3} (resp. $\tilde{A}_{i_1},\tilde{A}_{i_2},\tilde{A}_{i_3}$) be 3 blocks of the parallel class π_i (resp. $\tilde{\pi}_i$). Let B_{i_1},B_{i_2},B_{i_3} (resp. $\tilde{B}_{i_1},\tilde{B}_{i_2},\tilde{B}_{i_3}$) be the corresponding complementary blocks of A_{i_j} in D^* (resp. of $\tilde{A}_{i_j}$ in $\tilde{D}^*$).

Form 5 parallel classes of C as follows.

$$\begin{array}{ccccc} A_{i_1}\cup\tilde{A}_{i_1} & A_{i_2}\cup\tilde{A}_{i_2} & A_{i_3}\cup\tilde{A}_{i_3} & A_{i_1}\cup\tilde{A}_{i_2} & A_{i_1}\cup\tilde{A}_{i_3}\\ B_{i_1} & B_{i_2} & B_{i_3} & A_{i_2}\cup\tilde{A}_{i_3} & A_{i_2}\cup\tilde{A}_{i_1}\\ \tilde{B}_{i_1} & \tilde{B}_{i_2} & \tilde{B}_{i_3} & A_{i_3}\cup\tilde{A}_{i_1} & A_{i_3}\cup\tilde{A}_{i_2} \end{array}$$

It is clear that parallel classes of type I cover all the blocks of D^* and $\tilde{D}^*$.

Type II

For each pair (i,j), $i\neq j$ with $1\leq d(\pi_i,\pi_j)\leq s$, the nine blocks of the form $A\cup\tilde{A}$, where $A\in\pi_i$ and $\tilde{A}\in\tilde{\pi}_j$, are partitioned into 3 parallel classes as follows.

$$\begin{array}{ccc} A_{i_1}\cup\tilde{A}_{j_1} & A_{i_1}\cup\tilde{A}_{j_2} & A_{i_1}\cup\tilde{A}_{j_3}\\ A_{i_2}\cup\tilde{A}_{j_2} & A_{i_2}\cup\tilde{A}_{j_3} & A_{i_2}\cup\tilde{A}_{j_1}\\ A_{i_3}\cup\tilde{A}_{j_3} & A_{i_3}\cup\tilde{A}_{j_1} & A_{i_3}\cup\tilde{A}_{j_2} \end{array}$$

This shows that C is resolvable and has parameters $3-(6k,2k,\Theta)$, where $\Theta = \lambda(2k-1)(3k-2)/(k-2)$.

Now, if we repeat the construction above with C, we obtain a further resolvable $3-(12k,4k,\Theta(4k-1)(6k-2)/(2k-2))$ design. Continuing this procedure will provide a resolvable $3-(2^i 3k, 2^i k, \lambda\prod_{j=0}^{i-1}\theta_j)$ design after i steps of recursion, where $\theta_j = (2.2^j k-1)(3.2^j k-2)/(2^j k-2)$.

Thus we have proved the following result.

Theorem 2.10 *If a resolvable $3-(3k,k,\lambda)$ design with $k \geq 3$ exists, then a resolvable $3-(6k,2k,\lambda(2k-1)(3k-2)/(k-2))$ design exists. In particular, there exists a resolvable $3-(2^i 3k, 2^i k, \Theta)$ design for any $i \geq 1$, where $\Theta = \lambda\prod_{j=0}^{i-1}\theta_j$ and $\theta_j = (2.2^j k-1)(3.2^j k-2)/(2^j k-2)$.*

Remark 2.2 If $k=2$, then we start with the trivial 2-(6,2,1) design D. The complementary design D^* of D is the trivial 3-(6,4,3) design. The same argument in the proof of Theorem 2.10 shows that a resulting 3-(12,4,3) design C is resolvable. Therefore, the assumption $k \geq 3$ in Theorem 2.10 is not essential, it is made in order to avoid a zero division in the expression of θ_0.

Theorem 2.1 seems to be a crucial and powerful tool for constructing resolvable 3-designs. First of all, the case $v=2k$ provides the most known examples of resolvable 3-designs for infinitely many values of k, for instance Hadamard 3-designs. Furthermore, Theorem 2.6 finally asserts the abundancy of resolvable $3-(2k,k,\lambda)$ designs. Up to now very little was known about resolvable 3-designs with $v=3k$. Theorem 2.10 is therefore interesting, because it can be used to show (for example) that non-trivial resolvable 3-designs with $v=3k$ exist for infinitely many values of k. For any given value $k \geq 3$, applying Theorem 2.10 to the trivial resolvable $3-(3k,k,\binom{3k-3}{k-3})$ yields an infinite family of resolvable 3-designs with parameters $3-(2^i 3k, 2^i k, \binom{3k-3}{k-3}\prod_{j=0}^{i-1}\theta_j)$, where $\theta_j = (2.2^j k-1)(3.2^j k-2)/(2^j k-2)$. It should be mentioned that these designs are non-trivial for all $i \geq 1$.

We record this result in the following theorem.

Theorem 2.11 *For any integer $k \geq 3$ there is a resolvable 3-design with parameters $3-(2^i 3k, 2^i k, \binom{3k-3}{k-3}\prod_{j=0}^{i-1}\theta_j)$, where $\theta_j = (2.2^j k-1)(3.2^j k-2)/(2^j k-2)$, for any $i \geq 1$.*

Kramer and Magliveras [7] have shown the existence of 9 mutually disjoint copies of the $5-(24,8,1)$ Witt design. The blocks of the $5-(24,8,1)$ Witt design can be partitioned into 253 parallel classes each having three blocks, see for instance [8]. So we have a resolvable $3-(24,8,m21)$ design for $m=1,\ldots,9$. Starting Theorem 2.10 with each of these designs will provide a further family, which is presented in the following theorem.

Theorem 2.12 *For any $m=1,\ldots,9$ and $i \geq 1$, there is a resolvable 3-design with parameters $3-(2^i 24, 2^i 8, m21\prod_{j=0}^{i-1}\theta_j)$, where $\theta_j = (2^{j+4}-1)(3.2^{j+3}-2)/(2^{j+3}-2)$.*

Before we discuss the combination of Theorem 2.1 and Theorem 2.10, we consider the construction of a family of resolvable 3-designs for $k=8$ using Construction I.

Let D_i be a resolvable $3-(2^i 24,4,3)$ design, for all integer $i \geq 0$. For the existence of D_i, see [6]. Take a resolvable $3-(24,8,105)$ design for D_0^*. Starting with the pair (D_0, D_0^*) and applying Construction I repeatedly as shown above for the family in Theorem 2.5 we obtain a family of resolvable 3-designs with parameters $3-(2^i 24, 8, 21(2^{i-2}24-1))$. For instance,

D_1^* (resp. D_2^*) has parameters $3-(48,8,231)$ (resp. $3-(96,8,483)$). To our knowledge this family is unknown.

We record the result in the following theorem.

Theorem 2.13 *There exists a resolvable* $3-(2^i24,8,21(2^{i-2}24-1))$ *design for all integer* $i\geq 0$.

We now show how to combine Theorems 2.1 and 2.10 by presenting a family of resolvable 3-designs for $k=16$.

Let D_i be a resolvable $3-(2^i24,8,21(2^{i-2}24-1))$ design in Theorem 2.10. We start with D_0 as a 3-(24,8,105) design and D_0^* as a $3-(24,16,1050)$ design, the complement of D_0. Then the constructed design D_1^* has parameters $3-(48,16,5775)$ and is resolvable. Applying Construction I to the pair D_1, D_1^* yields a further resolvable design D_2^*. Continuing this way, the constructed design D_i^* is resolvable and has parameters $3-(2^i24,16,7.15(2^{i-2}24-1)(2^{i-3}24-1))$. To verify this, we need to check the divisibility condition for $\frac{\Lambda^{(i)}k(k-2)}{2\lambda^{(i)}(v_i-k)}$ and to compute $\Lambda^{(i)}$. Since $\Lambda^{(i)}=\frac{\Lambda^{(i-1)}k(v_{i-1}-2)}{2(v_{i-1}-k)}$ and $2\lambda^{(i)}=21.(v_{i-1}-2)$, we have $\frac{\Lambda^{(i)}k(k-2)}{2\lambda^{(i)}(v_i-k)}=\frac{\Lambda^{(i-1)}k(k-2)}{2\lambda^{(i-1)}(v_{i-1}-k)}=\cdots=\frac{\Lambda^{(0)}k(k-2)}{2\lambda^{(0)}(v_0-k)}=(2k-1)=15$. Hence, $\Lambda^{(i)}=15.2\lambda^{(i)}(v_i-k)/k(k-2)=7.15(2^{i-2}24-1)(2^{i-3}24-1))$ as desired.

We obtain the following result.

Theorem 2.14 *For any integer* $j\geq 0$*, there exists a resolvable* $3-(2^j48,16,7.15.(2^{j-2}48-1)(2^{j-3}48-1))$ *design.*

The construction of the family in Theorem 2.14 can be recursively carried out with respect to each given block size 2^n , $n\geq 3$. In this way we obtain a double infinite family of resolvable 3-designs. In the following, we sketch this procedure.

For each $n\geq 3$, set $k_n=2^n$ and $v_{n,j}=2^j3.2^n$.

Starting with the family of resolvable designs in Theorem 2.13: $3-(v_{3,j},k_3,\lambda^{3,j})$, where $\lambda^{3,j}=\frac{1}{2}(k_3-2)(k_3-1)(2^{j-2}.3.2^3-1)=\frac{1}{2}(k_3-2)(k_3-1)(v_{3,j-2}-1)$, we obtain a family of resolvable 3-designs in Theorem 2.14: $3-(v_{4,j},k_4,\lambda^{4,j})$ with $\lambda^{4,j}=\frac{1}{2}(k_4-2)(k_4-1)(v_{4,j-2}-1)(v_{4,j-3}-1)$, for all $j\geq 0$, by using Theorem 2.1 and 2.10, which will be called the combined procedure, or CP for short.

Now starting with the family: $3-(v_{4,j},k_4,\lambda^{4,j})$ and applying CP, we obtain a new family of resolvable designs: $3-(v_{5,j},k_5,\lambda^{5,j})$, with $\lambda^{5,j}=\frac{1}{2}(k_5-2)(k_5-1)(v_{5,j-2}-1)(v_{5,j-3}-1)(v_{5,j-4}-1)$, for all $j\geq 0$.

When repeating the application of CP to the new family just constructed, we will obtain for each $n\geq 3$ a family of resolvable 3-designs having parameters $3-(2^j.3.2^n,2^n,\lambda^{n,j})$, for all $j\geq 0$, where $\lambda^{n,j}=(2^{n-1}-1)(2^n-1)\prod_{i=2}^{n-1}(2^{j-i}.3.2^n-1))$.

The divisibility condition of Theorem 2.1 turns out to be $\lambda^{n+1,j}k_n(2k_n-2)/2\lambda^{n,j}(v_{n,j-1}-k_n)=2k_n-1$, by using the fact that $\lambda^{n+1,0}=\lambda^{n,0}.4.(2.k_n-1)/(k_n-1)$ and $\lambda^{n,j}=(2^{n-1}-1)(2^n-1)\prod_{i=2}^{n-1}(v_{n,j-i}-1))$.

We summarize this result in the following theorem.

Theorem 2.15 *Let* $n\geq 3$ *be an integer. Then there is a family* $\mathcal{F}_n$ *of resolvable 3-designs having parameters* $3-(2^j.3.2^n,2^n,(2^{n-1}-1)(2^n-1)\prod_{i=2}^{n-1}(2^{j-i}.3.2^n-1))$*, for all* $j\geq 0$.

3 Construction II

We have seen that Contruction I provides a class of 3-designs for which the size of blocks divides the number of points. In this section, we want to extend Construction I so that we are able to construct designs for which the number of points is not necessarily divisible by the size of the blocks.

Let $D_1 = (X, \mathcal{B}_1)$ be a resolvable $3-(v, k_1, \lambda)$ design and let $D_2 = (X, \mathcal{B}_2)$ be a resolvable $3-(v, k_2, \zeta)$ design with $3 \leq k_1 < k_2$ such that $\lambda\frac{(v-1)(v-2)}{(k_1-1)(k_1-2)} = \zeta\frac{(v-1)(v-2)}{(k_2-1)(k_2-2)}$, i.e. D_1 and D_2 have the same number of parallel classes. Let $\pi_1, \ldots, \pi_r$ (resp. $\Pi_1, \ldots, \Pi_r$) denote the r parallel classes of D_1 (resp. D_2).

Let $D_3 = (X, \mathcal{B}_3)$ be a $3-(v, k_1+k_2, \Lambda)$ design and let $\tilde{D}_i$ be a copy of D_i, $i = 1, 2, 3$, constructed on the point set $\tilde{X}$ with $X \cap \tilde{X} = \emptyset$.

Define blocks on the point set $X \cup \tilde{X}$ as follows:

I. blocks of D_3 and $\tilde{D}_3$;

II. blocks of the form $A \cup \tilde{B}$, where $A \in \pi_i$ and $\tilde{B} \in \tilde{\Pi}_j$ such that $\epsilon \leq d(\pi_i, \pi_j) \leq s$, $\epsilon = 0, 1$;

III. blocks of the form $\tilde{A} \cup B$, where $\tilde{A} \in \tilde{\pi}_i$ and $B \in \Pi_j$ such that $\epsilon \leq d(\pi_i, \pi_j) \leq s$, $\epsilon = 0, 1$.

Any 3 points $a, b, c \in X$ (resp. $\tilde{a}, \tilde{b}, \tilde{c} \in \tilde{X}$) are contained in, Λ blocks of type I, $(2s+1-\epsilon)\lambda\frac{v}{k_2}$ blocks of type II and $(2s+1-\epsilon)\zeta\frac{v}{k_1}$ blocks of type III. Thus they appear in $\Lambda + (2s+1-\epsilon)\lambda\frac{v}{k_2} + (2s+1-\epsilon)\zeta\frac{v}{k_1}$ blocks.

We need to compute the number of blocks containing 3 points of type $a, b, \tilde{c}$, where $a, b \in X$ and $\tilde{c} \in \tilde{X}$; the case for three points $\tilde{a}, \tilde{b}, c$ is similar.

Now a and b are contained in $\lambda\frac{v-2}{k_1-2}$ blocks of D_1 and $\tilde{c}$ is in exactly one block of each parallel class of $\tilde{D}_2$. So $a, b, \tilde{c}$ are in $(2s+1-\epsilon)\lambda\frac{v-2}{k_1-2}$ blocks of type II. Similarly, $a, b, \tilde{c}$ are in $(2s+1-\epsilon)\zeta\frac{v-2}{k_2-2}$ blocks of type III. Thus $a, b, \tilde{c}$ are in $(2s+1-\epsilon)\lambda\frac{v-2}{k_1-2} + (2s+1-\epsilon)\zeta\frac{v-2}{k_2-2}$ blocks.

These defined blocks will form a 3-design if

$$\Lambda + (2s+1-\epsilon)\lambda\frac{v}{k_2} + (2s+1-\epsilon)\zeta\frac{v}{k_1} = (2s+1-\epsilon)\lambda\frac{v-2}{k_1-2} + (2s+1-\epsilon)\zeta\frac{v-2}{k_2-2}$$

or

$$\Lambda = [\lambda\frac{v-2}{k_1-2} + \zeta\frac{v-2}{k_2-2} - (\lambda\frac{v}{k_2} + \zeta\frac{v}{k_1})](2s+1-\epsilon)$$

There are two cases:

Case A.

$$\lambda\frac{v-2}{k_1-2} + \zeta\frac{v-2}{k_2-2} - (\lambda\frac{v}{k_2} + \zeta\frac{v}{k_1}) = 0.$$

This implies $\Lambda = 0$ and the designs D_3 and $\tilde{D}_3$ are not needed in the construction. That means that the blocks of type II and III themselves form a design for $0 \leq s \leq \lfloor\frac{r}{2}\rfloor$. In this case, we can construct a $3-(2v, k_1+k_2, \Theta)$ design with $\Theta = m.(\lambda\frac{v-2}{k_1-2} + \zeta\frac{v-2}{k_2-2})$ for any $m = 1, \ldots, r$.

Case B.

$$\lambda\frac{v-2}{k_1-2} + \zeta\frac{v-2}{k_2-2} - (\lambda\frac{v}{k_2} + \zeta\frac{v}{k_1}) > 0.$$

Here the defined blocks form a design if

$$\Lambda/[\lambda\frac{v-2}{k_1-2}+\zeta\frac{v-2}{k_2-2}-(\lambda\frac{v}{k_2}+\zeta\frac{v}{k_1})]=\Omega$$

is a positive integer $\leq r$.

The parameters of the constructed design are $3-(2v,k_1+k_2,\Theta)$, where $\Theta=\Omega(\lambda\frac{v-2}{k_1-2}+\zeta\frac{v-2}{k_2-2})$.

We summarize Construction II in the following theorem.

Theorem 3.1 *Suppose that there exists a resolvable* $3-(v,k_1,\lambda)$ *design* D_1 *and a resolvable* $3-(v,k_2,\zeta)$ *design* D_2 *with* $3\leq k_1<k_2$ *such that* $\lambda\frac{(v-1)(v-2)}{(k_1-1)(k_1-2)}=\zeta\frac{(v-1)(v-2)}{(k_2-1)(k_2-2)}=r$.

(i) *If* $\lambda\frac{v-2}{k_1-2}+\zeta\frac{v-2}{k_2-2}-(\lambda\frac{v}{k_2}+\zeta\frac{v}{k_1})=0$, *then there is a* $3-(2v,k_1+k_2,\Theta)$ *design with* $\Theta=m(\lambda\frac{v-2}{k_1-2}+\zeta\frac{v-2}{k_2-2})$ *for any* $m=1,\ldots,r$.

(ii) *If* $\lambda\frac{v-2}{k_1-2}+\zeta\frac{v-2}{k_2-2}-(\lambda\frac{v}{k_2}+\zeta\frac{v}{k_1})>0$ *and if there is a* $3-(v,k_1+k_2,\Lambda)$ *design* D_3 *such that*

$$\Lambda/[\lambda\frac{v-2}{k_1-2}+\zeta\frac{v-2}{k_2-2}-(\lambda\frac{v}{k_2}+\zeta\frac{v}{k_1})] \quad = \quad \Omega \tag{1}$$

is a positive integer $\leq r$, *then there is a* $3-(2v,k_1+k_2,\Theta)$ *design with* $\Theta=\Omega(\lambda\frac{v-2}{k_1-2}+\zeta\frac{v-2}{k_2-2})$.

Remark 3.1 If $\lambda\frac{v-2}{k_1-2}+\zeta\frac{v-2}{k_2-2}-(\lambda\frac{v}{k_2}+\zeta\frac{v}{k_1})<0$, then no design can be constructed.

As a first example of Construction II, take a resolvable 3-(12,4,3) design as D_1 and the resolvable 3-(12,6,10) design in Theorem 2.8 as D_2. Take the trivial 3-(12,10,36) design as D_3. Then Construction II yields a 3-(24,10,360) design. The latter is indicated as unknown in [5], p.55.

Theorem 3.2 *A* $3-(24,10,360)$ *design exists.*

As a second example consider a resolvable 3-(18,6,35) design D_1 and a resolvable 3-(18,9,98) design D_2. Note that D_1 is obtained from Theorem 2.10 by using the trivial 3-(9,3,1) design and D_2 is from Theorem 2.8. We have $\lambda\frac{v-2}{k_1-2}+\zeta\frac{v-2}{k_2-2}-(\lambda\frac{v}{k_2}+\zeta\frac{v}{k_1})=0$, and so there is $3-(36,15,m364)$ design for any $m=1,\ldots,476$.

Theorem 3.3 *There is a* $3-(36,15,m364)$ *design for any* $m=1,\ldots,476$.

4 Construction III

In this section we present a further construction of 3-designs having block size not dividing the number of points.

Let $T=(X,\mathcal{B}_T)$ be a resolvable $3-(v,\ell,\lambda)$ design. Let $\pi_1,\ldots,\pi_r$ denote the r parallel classes of T where $r=\lambda\frac{(v-1)(v-2)}{(l-1)(l-2)}$. As before, define a distance between any two parallel classes π_i and π_j of T by $d(\pi_i,\pi_j)=\min\{|i-j|,r-|i-j|\}$.

Let $\tilde{T}=(\tilde{X},\mathcal{B}_{\tilde{T}})$ be a copy of T defined on $\tilde{X}$ with $X\cap\tilde{X}=\emptyset$. Let $D=(X,\mathcal{B}_D)$ be a $3-(v,k,\Lambda)$ design, such that $w=k-\ell\geq 3$. Let $\tilde{D}$ be a copy of D defined on $\tilde{X}$.

Further, let W be a $3-(\ell, w, \theta)$ design. We also assume that any two blocks of T have less than w points in common. This condition guarantees that the resulting design is simple; if this condition is removed then the constructed design may have repeated blocks.

Define blocks on the point set $X \cup \tilde{X}$ as follows:

I. blocks of D and $\tilde{D}$;

II. blocks of the form $B \cup \tilde{Z}$, where $B \in \pi_i$ and $\tilde{Z}$ is a block of the design W defined on the points of a block in $\tilde{\pi}_j$ with $\epsilon \leq d(\pi_i, \pi_j) \leq s$, $\epsilon = 0, 1$;

III. blocks of the form $\tilde{B} \cup Z$, where $\tilde{B} \in \tilde{\pi}_i$ and Z is a block of the design W defined on the points of a block in π_j with $\epsilon \leq d(\pi_i, \pi_j) \leq s$, $\epsilon = 0, 1$.

Let $\{x, y, z\}$ be three points in X.

- $\{x, y, z\}$ are on Λ blocks of type I.
- $\{x, y, z\}$ are on λ blocks of T distributed in λ parallel classes π_i. As there are $(2s+1-\epsilon)$ parallel classes π_j satisfying $\epsilon \leq d(\pi_i, \pi_j) \leq s$, and there are v/ℓ blocks in $\tilde{\pi}_j$, there are $(2s+1-\epsilon)\lambda\theta_0\frac{v}{\ell}$ choices for blocks of type II containing $\{x, y, z\}$, where θ_0 is the number of blocks of W.
- There are λ parallel classes π_i having a block containing $\{x, y, z\}$. In the copy of W defined on the points of that block, $\{x, y, z\}$ are in θ blocks Z. Thus there are $\lambda\theta$ choices for Z. Further, there are v/ℓ blocks $\tilde{B}$ in $\tilde{\pi}_j$ with $\epsilon \leq d(\tilde{\pi}_i, \tilde{\pi}_j) \leq s$, so there are $\lambda\theta(2s+1-\epsilon)\frac{v}{\ell}$ blocks of type III containing $\{x, y, z\}$.

Altogether, there are $\Lambda + \lambda(2s+1-\epsilon)\theta_0\frac{v}{\ell} + \lambda(2s+1-\epsilon)\theta\frac{v}{\ell}$ blocks containing $\{x, y, z\}$.

Let $\{x, y, \tilde{z}\}$ be three points with $x, y \in X$ and $\tilde{z} \in \tilde{X}$.

- Two points $\{x, y\}$ are in $\lambda\frac{v-2}{\ell-2}$ blocks of T distributed in $\lambda\frac{v-2}{\ell-2}$ parallel classes π_i. For each of these π_i, there are $(2s+1-\epsilon)$ choices for $\tilde{\pi}_j$ with $\epsilon \leq d(\pi_i, \pi_j) \leq s$, and in $\tilde{\pi}_j$ there is a unique block containing $\tilde{z}$, so $\tilde{z}$ is in θ_1 blocks $\tilde{Z}$ of W defined on that block, where θ_1 is the number of blocks containing a point in W. Hence, there are $\lambda\theta_1(2s+1-\epsilon)\frac{v-2}{\ell-2}$ blocks of type II containing $\{x, y, \tilde{z}\}$.
- Each of $\lambda\frac{v-2}{\ell-2}$ parallel classes π_i, for which $\{x, y\}$ are on a block B, gives θ_2 blocks Z containing $\{x, y\}$ in the copy of W defined on B, where θ_2 is the number of blocks containing a pair of points in W. Further, there is a unique block $\tilde{B}$ containing $\tilde{z}$ in $\tilde{\pi}_j$ with $\epsilon \leq d(\tilde{\pi}_i, \tilde{\pi}_j) \leq s$, so there are $(2s+1-\epsilon)\lambda\frac{v-2}{\ell-2}\theta_2$ blocks of type III containing $\{x, y, \tilde{z}\}$.

Therefore, $\{x, y, \tilde{z}\}$ are in $\lambda\theta_1(2s+1-\epsilon)\frac{v-2}{\ell-2} + (2s+1-\epsilon)\lambda\frac{v-2}{\ell-2}\theta_2$ blocks.

The blocks so constructed will form a 3-design if

$$\begin{aligned}\Lambda + \lambda(2s+1-\epsilon)\theta_0\frac{v}{\ell} + \lambda(2s+1-\epsilon)\theta\frac{v}{\ell} &= \lambda\theta_1(2s+1-\epsilon)\frac{v-2}{\ell-2} \\ &\quad +(2s+1-\epsilon)\lambda\frac{v-2}{\ell-2}\theta_2 \qquad (2)\end{aligned}$$

or equivalently,

$$\Lambda/\lambda\theta[\frac{v-2}{\ell-2}(\frac{\binom{\ell-1}{2}}{\binom{w-1}{2}}+\frac{\ell-2}{w-2})-\frac{v}{\ell}(\frac{\binom{\ell}{3}}{\binom{w}{3}}+1)]=(2s+1-\epsilon)$$

is an integer $\leq r$. And the resulting design has parameters $3-(2v,k,\Theta)$, where

$$\Theta=(2s+1-\epsilon)\lambda\theta\frac{v-2}{\ell-2}(\frac{\binom{\ell-1}{2}}{\binom{w-1}{2}}+\frac{\ell-2}{w-2}).$$

We summarize Construction III in the following theorem.

Theorem 4.1 *Suppose that there exists a resolvable $3-(v,\ell,\lambda)$ design T and a $3-(v,k,\Lambda)$ design D with $w=k-\ell\geq 3$, $k\leq 2\ell$, and $|A\cap B|\leq w-1$ for any two distinct blocks A and B of T. Suppose that there is a $3-(\ell,w,\theta)$ design W such that*

$$\Lambda/\lambda\theta[\frac{v-2}{\ell-2}(\frac{\binom{\ell-1}{2}}{\binom{w-1}{2}}+\frac{\ell-2}{w-2})-\frac{v}{\ell}(\frac{\binom{\ell}{3}}{\binom{w}{3}}+1)]=\Omega$$

is an integer $\leq r$, where r is the number of parallel classes of T. Then there exists a $3-(2v,k,\Theta)$ design C, where $\Theta=\Omega\lambda\theta\frac{v-2}{\ell-2}(\frac{\binom{\ell-1}{2}}{\binom{w-1}{2}}+\frac{\ell-2}{w-2})$.

As an application of Theorem 4.1 we have the following Corollary.

Corollary 4.2 *If there exists a $3-(4n,7,\Lambda)$ design for $n\equiv 4,8 \pmod{12}$ such that $5(n-1)|\Lambda$ and $\Lambda\leq 5(n-1)\binom{4n-1}{2}/3$, then there exists a $3-(8n,7,\Lambda\frac{2n-1}{n-1})$ design.*

Proof. Take a resolvable $3-(4n,4,1)$ design as T and the trivial $3-(4,3,1)$ design as W for Theorem 4.1. □

An example derived from Theorem 4.1 is as follows. Let T be the resolvable 3-(24,8,21) design, which is the Witt system 5-(24,8,1), and let D be a $3-(24,15,m5.7.13)$ design, which is the complementary design of a $3-(24,9,m84)$ design with $m\in\{1,\ldots,101\}$. Let take W to be the trivial 3-(8,7,5) design. It follows that $\Omega=\frac{5}{2}m$ is an integer if $m=2n$. In this case Theorem 4.1 yields a $3-(48,15,n5.7.11.13)$ design. It is known that a $3-(24,9,m84)$ exists for all even values of m, see [5], p.55, so we have the following theorem.

Theorem 4.3 *There is a $3-(48,15,n5.7.11.13)$ design for $n=1,\ldots,50$.*

5 Special Constructions for $k=5,7,8$

In this section we present three special constructions for 3-designs with block sizes 5,7,8.

5.1 A construction for $k=5$

Let $D=(X,\mathcal{B})$ be a $3-(2n,5,\lambda)$ design. And let $\tilde{D}=(\tilde{X},\tilde{\mathcal{B}})$ be a copy of D with $X\cap\tilde{X}=\emptyset$. Let T be the resolvable $2-(2n,2,1)$ design defined on X. Let $T_1,\ldots,T_{2n-1}$ denote the $2n-1$ parallel classes of T. Define blocks for a $3-(4n,5,\Lambda)$ design on the point set $X\cup\tilde{X}$ as follows:

I. blocks of D (resp. blocks of $\tilde{D}$);

II. blocks of the form $\{a, b, c, \tilde{d}, \tilde{e}\}$ (resp.$\{\tilde{a}, \tilde{b}, \tilde{c}, d, e\}$), where $\{a, b\} \in T_h$, $\{b, c\} \in T_i$, $\{c, a\} \in T_j$, $\{\tilde{d}, \tilde{e}\} \in \tilde{T}_\ell$ and $\ell \in \{h, i, j\}$.

Any three points $a, b, c \in X$ (resp. $\tilde{a}, \tilde{b}, \tilde{c} \in \tilde{X}$) are contained in λ blocks of type I and $3n$ blocks of type II.

Any three points $a, b, \tilde{d}$ with $a, b \in X$ and $\tilde{d} \in \tilde{X}$, (resp. $\tilde{a}, \tilde{b}, d$) are contained in

- $3(2n-2)$ blocks of type $\{a, b, c, \tilde{d}, \tilde{e}\}$: there are $(2n-2)$ choices for c and for each such c there are 3 possibilities for $\tilde{e}$ such that $\{\tilde{d}, \tilde{e}\} \in \{\tilde{T}_h, \tilde{T}_i, \tilde{T}_j\}$;
- $(3n-3)$ blocks of type $\{\tilde{e}, \tilde{d}, \tilde{c}, a, b\}$: if $\{a, b\} \in T_h$, then there are $(2n-1)$ choices for $\tilde{c}$, exactly one of them gives $\{\tilde{c}, \tilde{d}\} \in \tilde{T}_h$ and hence there are $(2n-2)$ possible choices for $\tilde{e}$; from the remaining $(2n-2)$ possible choices for $\tilde{c}$ we have $\{\tilde{c}, \tilde{d}\} \in \tilde{T}_i \neq \tilde{T}_h$ and $\tilde{e}$ has to be chosen such that $\{\tilde{c}, \tilde{e}\} \in \tilde{T}_h$, so there are $(n-1)$ choices for the pair $\{\tilde{c}, \tilde{e}\}$ as a block in $\tilde{T}_h$.

In summary, there are $3(2n-2)+(3n-3) = 9(n-1)$ blocks containing $a, b, \tilde{d}$. The blocks so defined will form a 3-design if and only if $\lambda + 3n = 9(n-1)$, or equivalently $\lambda = 6n-9$. The design constructed will have parameters $3-(4n, 5, 9(n-1))$. Hence, we have the following theorem.

Theorem 5.1 *If there is a* $3-(2n, 5, 6n-9)$ *design, then there is a* $3-(4n, 5, 9(n-1))$ *design.*

Examples 5.1 As an application, Theorem 5.1 shows the existence of a 3-(36,5,72) and a 3-(44,5,90) design since a 3-(18,5,45) and a 3-(22,5,57) design exist.

Remark 5.1 In the Driessen construction [4], p.87, D is the trivial $3-(2n, 5, \binom{2n-3}{2})$ design. In this case, the only value n for which a 3-design can be constructed is $n = 5$, and the design obtained has parameters 3-(20,5,36).

5.2 A construction for $k = 7$

Let $D = (X, \mathcal{B})$ be a $3-(3n, 7, \lambda)$ design. And let $\tilde{D} = (\tilde{X}, \tilde{\mathcal{B}})$ be a copy of D with $X \cap \tilde{X} = \emptyset$. Let T be the resolvable $3-(3n, 3, 1)$ design defined on X. Denote by $T_1, \ldots, T_r$ the parallel classes of T, where $r = \binom{3n-1}{2}$. Define blocks for a $3-(6n, 7, \Lambda)$ design on the point set $X \cup \tilde{X}$ as follows:

I. blocks of D (resp. blocks of $\tilde{D}$);

II. sets of the form $\{a, b, c, d, \tilde{e}, \tilde{f}, \tilde{g}\}$ (resp.$\{\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, e, f, g\}$), where $\{a, b, c\} \in T_{i_1}$, $\{b, c, d\} \in T_{i_2}$, $\{c, d, a\} \in T_{i_3}$, $\{d, a, b\} \in T_{i_4}$, and $\{\tilde{e}, \tilde{f}, \tilde{g}\} \in \tilde{T}_j$ and $j \in \{i_1, i_2, i_3, i_4\}$.

Any three points $a, b, c \in X$ (resp. $\tilde{a}, \tilde{b}, \tilde{c} \in \tilde{X}$) are contained in

- λ blocks of type I;
- $4n(3n-3)$ blocks of form $\{a, b, c, d, \tilde{e}, \tilde{f}, \tilde{g}\}$: there are $(3n-3)$ possible choices for d and each such a choice determines 4 parallel classes T_{i_1}, T_{i_2}, T_{i_3}, and T_{i_4}, the points $\tilde{e}, \tilde{f}, \tilde{g}$ have to be chosen such that they form a block of $\tilde{T}_{i_j}$, $j = 1, 2, 3, 4$, so there are $4n(3n-3)$ blocks containing $\{a, b, c\}$;

- $n(3n-3)$ blocks of form $\{\tilde{g}, \tilde{f}, \tilde{e}, \tilde{d}, a, b, c\}$: if $\{a,b,c\} \in T_j$, then some 3 points of $\{\tilde{g}, \tilde{f}, \tilde{e}, \tilde{d}\}$ must be a block in $\tilde{T}_j$, so there are n possible choices for those three points, the fourth point can be chosen in $(3n-3)$ ways; thus there are $n(3n-3)$ blocks containing $\{a,b,c\}$.

Hence there are $\lambda + 4n(3n-3) + n(3n-3) = \lambda + 5n(3n-3)$ blocks of type I and II containing a, b, c.
Any three points $a, b, \tilde{e}$ with $a, b \in X$ and $\tilde{e} \in \tilde{X}$ are contained in

- $6(3n-2)(n-1)$ blocks of form $\{a, b, c, d, \tilde{e}, \tilde{f}, \tilde{g}\}$: there are $\binom{3n-2}{2}$ possible choices for a pair $\{c, d\}$, each choice determines 4 parallel classes, and two points $\tilde{f}, \tilde{g}$ have to be chosen so that $\{\tilde{e}, \tilde{f}, \tilde{g}\}$ is a block in one of these 4 parallel classes, thus there are 4 choices for $\{\tilde{e}, \tilde{f}, \tilde{g}\}$; altogether we have $4\binom{3n-2}{2} = 6(3n-2)(n-1)$ blocks of form $\{a, b, c, d, \tilde{e}, \tilde{f}, \tilde{g}\}$ containing $a, b, \tilde{e}$;

- $4(3n-2)(n-1)$ blocks of form $\{\tilde{g}, \tilde{f}, \tilde{e}, \tilde{d}, a, b, c\}$: for each of $(3n-2)$ choices for $c \neq a, b$ denote by T_j the parallel class containing $\{a, b, c\}$ as a block; there is exactly one block of $\tilde{T}_j$ containing $\tilde{e}$ and also two other points, say $\tilde{d}$, $\tilde{f}$; the last point $\tilde{g}$ can be chosen in $(3n-3)$ different ways, this gives $(3n-2)(3n-3)$ blocks; on the other hand, for any of $(3n-2)$ choices for c, there are $(n-1)$ blocks of the form $\{\tilde{d}, \tilde{f}, \tilde{g}\}$ in $\tilde{T}_j$, so this gives $(n-1)(3n-2)$ blocks; altogether there are $(3n-2)(3n-3) + (n-1)(3n-2) = 4(n-1)(3n-2)$ blocks of form $\{\tilde{g}, \tilde{f}, \tilde{e}, \tilde{d}, a, b, c\}$ containing $a, b, \tilde{e}$.

The blocks constructed on $X \cup \tilde{X}$ will form a design if any 3 points of the form a, b, c and $a, b, \tilde{e}$ are contained in the same number of blocks, i.e. if the condition $\lambda + 5n(3n-3) = 6(n-1)(3n-2) + 4(n-1)(3n-2)$ is satisfied. Hence $\lambda = 5(n-1)(3n-4)$. So, any three points of the constructed design are contained in $\Lambda = 6(n-1)(3n-2) + 4(n-1)(3n-2) = 10(n-1)(3n-2)$ blocks. Therefore we have the following theorem.

Theorem 5.2 *If there is a* $3-(3n, 7, 5(n-1)(3n-4))$ *design, then there is a* $3-(6n, 7, 10(n-1)(3n-2))$ *design for all* $n \geq 0$.

As examples we see that if a 3-(21,7,510) (resp. 3-(30,7,1170)) design exists then there exists a 3-(42,7,1140) (resp. 3-(60,7,2520)) design.

5.3 A construction for $k = 8$

In the same vein as the construction for $k = 7$, we may also construct designs for $k = 8$ when using the trivial $3-(3n, 3, 1)$ design.

Let $D = (X, \mathcal{B})$ be a $3-(3n, 8, \lambda)$ design. Let $\tilde{D} = (\tilde{X}, \tilde{\mathcal{B}})$ be a copy of D with $X \cap \tilde{X} = \emptyset$. Again, let T be the resolvable $3-(3n, 3, 1)$ design defined on X. Denote by $T_1, \ldots, T_r$ the parallel classes of T, where $r = \binom{3n-1}{2}$. Define blocks for a $3-(6n, 8, \Lambda)$ design on the point set $X \cup \tilde{X}$ as follows:

I. blocks of D (resp. blocks of $\tilde{D}$);

II. blocks of the form $\{a, b, c, d, e, \tilde{f}, \tilde{g}, \tilde{h}\}$ (resp.$\{\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{e}, f, g, h\}$), having the property that if $\{\tilde{f}, \tilde{g}, \tilde{h}\} \in \tilde{T}_i$ then there are three points $\{x, y, z\} \subseteq \{a, b, c, d, e\}$ with $\{x, y, z\} \in T_i$.

Any three points $a, b, c \in X$ (resp. $\tilde{a}, \tilde{b}, \tilde{c} \in \tilde{X}$) are contained in

- λ blocks of type I;
- $10n\binom{3n-3}{2}$ blocks of form $\{a,b,c,d,e,\tilde{f},\tilde{g},\tilde{h}\}$: there are $\binom{3n-3}{2}$ possible choices for a pair $\{d,e\}$ and each choice determines 10 parallel classes T_{i_j}, $j=1,\ldots,10$, each of these classes contains exactly one 3-subset of $\{a,b,c,d,e\}$; points $\tilde{f},\tilde{g},\tilde{h}$ have to be chosen such that they form a block of $\tilde{T}_{i_j}$, this yields $10n\binom{3n-3}{2}$ blocks containing $\{a,b,c\}$;
- $n\binom{3n-3}{2}$ blocks of form $\{\tilde{h},\tilde{g},\tilde{f},\tilde{e},\tilde{d},a,b,c\}$: if $\{a,b,c\}\in T_i$, then some 3 points of $\{\tilde{h},\tilde{g},\tilde{f},\tilde{e},\tilde{d}\}$ must be as a block in $\tilde{T}_i$, so there are n possible choices for those 3 points, the other two points of $\{\tilde{h},\tilde{g},\tilde{f},\tilde{e},\tilde{d}\}$ can be chosen $\binom{3n-3}{2}$ ways; this yields $n\binom{3n-3}{2}$ blocks containing $\{a,b,c\}$.

Hence there are $\lambda+10n\binom{3n-3}{2}+n\binom{3n-3}{2}=\lambda+11n\binom{3n-3}{2}$ blocks of type I and II containing a,b,c.
Any three points $a,b,\tilde{f}$ with $a,b\in X$ and $\tilde{f}\in\tilde{X}$ are contained in

- $10\binom{3n-2}{3}$ blocks of form $\{a,b,c,d,e,\tilde{f},\tilde{g},\tilde{h}\}$: there are $\binom{3n-2}{3}$ possible choices for a triple $\{c,d,e\}$; five points $\{a,b,c,d,e\}$ determine 10 parallel classes, T_{i_j}, $j=1,\ldots,10$, each of these classes contains exactly one 3-subset of $\{a,b,c,d,e\}$; and points $\tilde{g},\tilde{h}$ have to be chosen so that $\{\tilde{f},\tilde{g},\tilde{h}\}$ is a block of T_{i_j}, so there are 10 choices for $\{\tilde{f},\tilde{g},\tilde{h}\}$, this gives $10\binom{3n-2}{3}$ blocks containing $a,b,\tilde{f}$;
- $5\binom{3n-2}{3}$ blocks of form $\{\tilde{h},\tilde{g},\tilde{f},\tilde{e},\tilde{d},a,b,c\}$: for each of $(3n-2)$ choices for $c\neq a,b$ let T_i be the parallel class containing $\{a,b,c\}$ as a block; there is exactly one block of $\tilde{T}_i$ containing $\tilde{f}$ and also two other points, say $\tilde{d}$, $\tilde{e}$; the other two points $\tilde{g}$ and $\tilde{h}$ can be chosen in $\binom{3n-3}{2}$ different ways, this yields $(3n-2)\binom{3n-3}{2}=3\binom{3n-2}{3}$ blocks containing $a,b,\tilde{f}$; on the other hand, for any of $(3n-2)$ choices for c, there are $(n-1)$ blocks of form $\{\tilde{x},\tilde{y},\tilde{z}\}$ in $\tilde{T}_i$, where $\{\tilde{x},\tilde{y},\tilde{z}\}\subseteq\{\tilde{h},\tilde{g},\tilde{e},\tilde{d}\}$, and there are $(3n-4)$ possible choices for another point of $\{\tilde{h},\tilde{g},\tilde{e},\tilde{d}\}$, this yields $(3n-2)(n-1)(3n-4)=2\binom{3n-2}{3}$ blocks containing $a,b,\tilde{f}$; altogether there are $3\binom{3n-2}{3}+2\binom{3n-2}{3}=5\binom{3n-2}{3}$ blocks of form $\{\tilde{h},\tilde{g},\tilde{f},\tilde{e},\tilde{d},a,b,c\}$ containing $a,b,\tilde{f}$.

The blocks constructed on $X\cup\tilde{X}$ will form a design if any 3 points of the form a,b,c and $a,b,\tilde{f}$ are contained in the same number of blocks, i.e. if the condition $\lambda+11n\binom{3n-3}{2}=10\binom{3n-2}{3}+5\binom{3n-2}{3}$ is satisfied. Hence $\lambda=(2n-5)(3n-3)(3n-4)$.

So, any three points of the design constructed are contained in $\Lambda=10\binom{3n-2}{3}+5\binom{3n-2}{3}=15\binom{3n-2}{3}$ blocks. Therefore we have the following theorem.

Theorem 5.3 *If there is a* $3-(3n,8,(2n-5)(3n-3)(3n-4))$ *design, then there is a* $3-(6n,8,15\binom{3n-2}{3})$ *design for all* $n\geq 0$.

Examples 5.2 There is a 3-(36,8,8400) (resp. 3-(48,8,23100)) design since there is a 3-(18,8,1470) (resp. 3-(24,8,4620)) design.

6 Appendix

The following table contains a list of parameters for 3-designs constructed from the recursive methods of the paper.

	Parameters	Comments	Theorems
1.	$3-(30,6,m5)$, $m=13,26,39,52,65$		Thm. 2.2
2.	$3-(40,8,m63)$, $m=1,\ldots,8$		Thm. 2.3
3.	$3-(2^{n+1}+2,6,5(2^n-1))$, odd $n\geq 5$		Thm. 2.4
4.	$3-(2^i20,8,7(2^{i-2}20-1))$, $i\geq 0$		Thm. 2.5
5.	$3-(2^i28,8,7(2^{i-2}28-1))$, $i\geq 0$		Thm. 2.5
6.	$3-(24,6,m10)$, $m=1,\ldots,14$	resolvable	Thm. 2.9
7.	$3-(32,8,m7)$, $m=1,\ldots,35$	resolvable	Thm. 2.9
8.	$3-(2^i3k,2^ik,\binom{3k-3}{k-3}\prod_{j=0}^{i-1}\theta_j)$,		
	$\theta_j=(2.2^jk-1)(3.2^jk-2)/(2^jk-2)$, $i\geq 1$	resolvable	Thm. 2.11
9.	$3-(2^i24,2^i8,m21\prod_{j=0}^{i-1}\theta_j)$, $m=1,\ldots,9$, $i\geq 1$,		
	$\theta_j=(2^{j+4}-1)(3.2^{j+3}-2)/(2^{j+3}-2)$	resolvable	Thm. 2.12
10.	$3-(2^i24,8,21(2^{i-2}24-1))$, $i\geq 0$	resolvable	Thm. 2.13
11.	$3-(2^j48,16,7.15.(2^{j-2}48-1)(2^{j-3}48-1))$, $j\geq 0$	resolvable	Thm. 2.14
12.	$3-(2^j.3.2^n,2^n,(2^{n-1}-1)(2^n-1)\prod_{i=2}^{n-1}(2^{j-i}.3.2^n-1))$,		
	$j\geq 0$, for any $n\geq 3$	resolvable	Thm. 2.15
13.	3-(24,10,360)		Thm. 3.2
14.	$3-(36,15,m364)$, $m=1,\ldots,476$		Thm. 3.3
15.	$3-(48,15,m5.7.11.13)$, $m=1,\ldots,50$		Thm. 4.3

Remark 6.1 Families 10 and 11 in the table are special cases of family 12 with $n=3$ and 4.

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References

[1] W. O. Alltop, Extending t-designs, *J. Combin. Theory A* **12**, 177–186 (1975).

[2] Z. Baranyai, On the factorization of the complete uniform hypergraph, Proc. Erdös-Colloquium Keszthely 1973, p.91–108, North-Holland, Amsterdam.

[3] J. Bierbrauer, A new family of 4-designs, *Graphs Comb.* **11**, 209–211 (1995).

[4] L. H. M. E. Driessen, t-designs, $t \geq 3$, Technical Report, Department of Mathematics, Technische Hogeschool Eindhoven, The Netherlands, 1978.

[5] C. J. Colbourn and J. H. Dinitz, Eds. *The CRC Handbook of Combinatorial Designs*, CRC Press (1996).

[6] D. Jungnickel and S. A. Vanstone, On resolvable designs $S_3(3;4,v)$, *J. Combin. Theory A* **43**, 334–337 (1986).

[7] E. S. Kramer and S. S. Magliveras, Some mutually disjoint Steiner systems, *J. Combin. Theory A* **17**, 39–43 (1974).

[8] E. S. Kramer, S. S. Magliveras and D. M. Mesner, Some resolutions of S(5,8,24), *J. Combin. Theory A* **29**, 166–173 (1980).

[9] E. Witt, Über Steinersche Systeme, *Abh. Math. Sem. Hamburg* **12**, 265–275 (1938).