Roux-type Constructions for Covering Arrays of Strengths Three and Four

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Abstract

A *covering array* $CA(N; t, k, v)$ is an $N \times k$ array such that every $N \times t$ sub-array contains all t-tuples from v symbols *at least* once, where t is the *strength* of the array. Covering arrays are used to generate software test suites to cover all t-sets of component interactions. Recursive constructions for covering arrays of strengths 3 and 4 are developed, generalizing many "Roux-type" constructions. A numerical comparison with current construction techniques is given through existence tables for covering arrays.

1 Introduction

A *covering array* $\mathsf{CA}(N; t, k, v)$ is an $N \times k$ array such that every $N \times t$ sub-array contains all t-tuples from v symbols *at least* once, where t is the *strength* of the array. When 'at least' is replaced by 'exactly', this defines an *orthogonal array* [19]. We use the notation $\mathsf{OA}(N; t, k, v)$. Often we refer to a t-covering array to indicate some $\mathsf{CA}(N; t, k, v)$. We denote by $\mathsf{CAN}(t, k, v)$ the minimum N for which a $\mathsf{CA}(N; t, k, v)$ exists. The determination of $\mathsf{CAN}(t, k, v)$ has been the subject of much research; see [8, 12, 17, 18] for survey material. However, only in the case of $\mathsf{CAN}(2, k, 2)$ is an exact determination known (see [12]). In part the interest arises from applications in software testing [11], but other applications in which experimental factors interact avail themselves of covering arrays as well [12, 17].

We outline the approaches taken for strength $t = 2$, but refer to [12] for a more detailed survey. When the number of factors is "small", numerous direct constructions have been developed. Some exploit the known structure of orthogonal arrays arising from the finite field, but most have a computational component. A range of methods have been applied, including greedy methods

[11], tabu search [25], simulated annealing [9], and constraint satisfaction [20]. Assuming that the covering array admits an automorphism can reduce the computational difficulty substantially [24].

At the other extreme, when the number of factors k goes to infinity, asymptotic methods have been applied; see [16], for example. In practice, this leaves a wide range of values of k for which no useful information can be deduced. Computational methods become infeasible, and asymptotic analysis does not apply, within this range. Hence there has been substantial interest in recursive ("product") constructions to make large covering arrays from smaller ones. Currently, the most general recursive constructions for strength two appear in [15].

This pattern is repeated for strength $t > 2$. The larger the strength, the more limited is our ability to obtain computational results for small numbers of factors. For strength three, powerful heuristic search such as simulated annealing [10] and tabu search [25] are still effective, but for larger strengths their current applications are quite restricted. Consequently, imposing larger automorphism groups to accelerate the search has proved effective in some cases [7, 8]. More recently, Sherwood *et al.* [27] developed a "permutation vector" representation for certain covering arrays. In conjunction with tabu search, Walker and Colbourn [33] produce many coverings arrays for strengths between 3 and 7.

Despite current limitations in producing t-covering arrays with a small number of factors, recursive constructions have proved to be effective in making arrays for larger numbers of factors. Roux [26] pioneered a conceptually simple recursive construction for strength $t = 3$ that has been substantially generalized for strength 3 [8, 10], strength 4 [17, 18, 23], and strength t in general [22, 23]. In this paper, we improve the recursion for strength 3, and we generalize and unify the Roux-type recursions for strength 4. We then recall related recursions using Turán families and perfect hash families in §5, and using this current census of known constructions we present current existence tables for covering arrays of strengths 3 and 4.

2 Definitions and Preliminaries

Let Γ be a group of order v, with $\odot$ as its binary operation. A $(v, k; \lambda)$-*difference matrix* $\mathsf{D} = (d_{ij})$ over Γ is a $v\lambda \times k$ matrix $\mathsf{D} = (d_{\ell,i})$ with entries from Γ, so that for each $1 \leq i < j \leq k$, the set $\{d_{\ell,i} \odot d_{\ell,j}^{-1} : 1 \leq \ell \leq v\lambda\}$ contains every element of Γ λ times. When Γ is abelian, additive notation is used, so that difference $d_{\ell,i} - d_{\ell,j}$ is employed. (Often in the literature the transpose of this definition is used.)

A t-*difference covering array* $\mathsf{D} = (d_{ij})$ over Γ, denoted by $\mathsf{DCA}(N, \Gamma; t, k, v)$, is an $N \times k$ array with entries from Γ having the property that for any t distinct columns $j_1, j_2, \ldots, j_t$, the set $\{(d_{i,j_1} \odot d_{i,j_2}^{-1}, d_{i,j_1} \odot d_{i,j_3}^{-1}, \ldots, d_{i,j_1} \odot d_{i,j_t}^{-1}) : 1 \leq i \leq N\}$ contains every nonzero $(t-1)$-tuple over Γ at least once. When $\Gamma = \mathbb{Z}_v$ we omit it from the notation. We denote by $\mathsf{DCAN}(t, k, v)$ the minimum N for which a $\mathsf{DCA}(N; t, k, v)$ exists.

A *covering ordered design* $\mathsf{COD}(N; t, k, v)$ is an $N \times k$ array such that every $N \times t$ sub-array contains all non-constant t-tuples from v symbols *at least* once. We denote by $\mathsf{CODN}(t, k, v)$ the minimum N for which a $\mathsf{COD}(N; t, k, v)$ exists.

A $\mathsf{QCA}(N; k, \ell, v)$ is an $N \times k\ell$ array with columns indexed by ordered pairs from $\{1, \ldots, k\} \times \{1, \ldots, \ell\}$, in which whenever $1 \leq i < j \leq k$ and $1 \leq a < b \leq \ell$, the $N \times 4$ subarray indexed by the four columns $(i, a), (i, b), (j, b), (j, a)$ contains every 4-tuple (x, y, z, t) with $x - t \not\equiv y - z \pmod{v}$ at least once. $\mathsf{QCAN}(k, \ell, v)$ denotes the minimum number of rows in such an array.

We recall two general results.

Theorem 2.1 *[19] When $v \geq 2$ is a prime power then an* $\mathsf{OA}(v^t; t, v, v+1)$ *exists whenever* $v \geq t-1 \geq 0$.

Theorem 2.2 *[14] The multiplication table for the finite field $\mathbb{F}_v$ is a $(v, v; 1)$-difference matrix over* $\mathrm{EA}(v)$.

In order to simplify the presentation later, we establish a basic result:

Theorem 2.3 $\mathsf{CAN}(2, k, vw) \leq min \left\{ \begin{array}{l} \mathsf{CAN}(2, k, v)\mathsf{CAN}(2, v, w) + v\mathsf{CODN}(2, k, w) \\ \mathsf{CODN}(2, k, v)\mathsf{CAN}(2, v, w) + v\mathsf{CAN}(2, k, w) \end{array} \right.$.

Proof. We prove the first statement; the second is similar. Suppose that there exist A a $\mathsf{CA}(N_A; 2, k, v)$, B a $\mathsf{CA}(N_B; 2, v, w)$, and C a $\mathsf{COD}(N_C; 2, k, w)$.

We produce a $\mathsf{CA}(N'; 2, k, vw)$ D where $N' = N_A N_B + v N_C$. D is formed by vertically juxtaposing arrays E of size $N_A N_B$ and $F^0, \ldots, \mathsf{F}^{v-1}$ each of size N_C.

We refer to elements of D as ordered pairs (a, b) where $0 \leq a < v$ and $0 \leq b < w$. There are vw such elements.

Define array E as follows. Replace each element i from A with a column of length N_B whose jth entry is (i, σ) where σ is the jth entry of the ith column of B.

Define array F^ℓ to be the result of replacing every entry σ of array C by (ℓ, σ). Then D has N' rows. We now verify that it is a $\mathsf{CA}(N'; 2, k, vw)$.

Consider columns i and j of D to verify the presence of the pair (r, x) in column i and (s, y) in column j.

If $r \neq s$, look in E. There is a row in A that covers the pair (r, s) in columns (i, j). We look at the expansion of this pair from A into E. Since there is also a row in B that covers the pair (x, y), say in row n, and since the rth and sth columns of B are distinct, the nth row of the expansion contains the required pair. Similarly if $r = s$ and $x = y$, there is a row in A that covers the pair (r, r) and all pairs are covered in the expansion into E provided that $x = y$.

It remains to treat the case when $r = s$ but $x \neq y$, i.e. the pairs sought are of the form (r, x) and (r, y). For these we consider F^r. Since $x \neq y$, the pair (x, y) is covered in C. So, the pair $(r, x), (r, y)$ is covered in F^r. ■

Corollary 2.4 *For v a prime power,*

$$\mathsf{CAN}(2, k, v^2) \leq min \left\{ \begin{array}{l} v^2\mathsf{CAN}(2, k, v) + v\mathsf{CODN}(2, k, v) \\ v^2\mathsf{CODN}(2, k, v) + v\mathsf{CAN}(2, k, v) \end{array} \right\} \leq (v^2 + v)\mathsf{CAN}(2, k, v) - v^2.$$

Proof. $\mathsf{CODN}(2, k, v) \leq \mathsf{CAN}(2, k, v) - 1$. ■

Theorem 2.5 $\mathsf{CODN}(2, k, vw) \leq \mathsf{CODN}(2, k, v)\mathsf{CODN}(2, v, w) + v\mathsf{CODN}(2, k, w)$.

Proof. This parallels the proof of Theorem 2.3 closely. ■

For large k, these improve upon the simple "composition" of covering arrays that establishes that $\mathsf{CAN}(2, k, vw) \leq \mathsf{CAN}(2, k, v)\mathsf{CAN}(2, k, w)$.

3 Strength Three

In [28], a theorem from Roux's Ph.D. dissertation [26] is presented.

Theorem 3.1 $\mathsf{CAN}(3, 2k, 2) \leq \mathsf{CAN}(3, k, 2) + \mathsf{CAN}(2, k, 2)$.

Proof. To construct a $\mathsf{CA}(3, 2k, 2)$, we begin by placing two $\mathsf{CA}(N_3, 3, k, 2)$s side by side. We now have a $N_3 \times 2k$ array. If one chooses any three columns whose indices are distinct modulo k, then all triples are covered. The remaining selection consists of a column x from among the first k, its copy among the second k, and a further column y. When the two columns whose indices agree modulo k share the same value, such a triple is also covered. The remaining triples are handled by appending two $\mathsf{CA}(N_2, 2, k, 2)$s side by side, the second being the bit complement of the first. Therefore if we choose two distinct columns from one half, we choose the bit complement of one of these, thereby handling all remaining triples. This gives a covering array of size $N_2 + N_3$. ■

Chateauneuf and Kreher [8] prove a generalization:

Theorem 3.2 $\mathsf{CAN}(3, 2k, v) \leq \mathsf{CAN}(3, k, v) + (v-1)\mathsf{CAN}(2, k, v)$.

Cohen, Colbourn, and Ling [10] generalize to permit the number of factors to be multiplied by $\ell \geq 2$ rather than two.

Theorem 3.3 *[10]* $\mathsf{CAN}(3, k\ell, v) \leq \mathsf{CAN}(3, k, v) + \mathsf{CAN}(3, \ell, v) + \mathsf{CAN}(2, \ell, v) \times \mathsf{DCAN}(2, k, v)$.

Here we establish a different generalization of the Roux construction for strength three.

Theorem 3.4 *For any prime power $v \geq 3$*

$$\mathsf{CAN}(3, vk, v) \leq \mathsf{CAN}(3, k, v) + (v-1)\mathsf{CAN}(2, k, v) + v^3 - v^2$$

Proof. Suppose that C_3 is a $\mathsf{CA}(N_3; 3, k, v)$ and C_2 is a $\mathsf{CA}(N_2; 2, k, v)$. Suppose that D is the $(v-1) \times v$ array obtained by removing the first row from the difference matrix in Theorem 2.2. Then $d_{i,j} = i \times j$ for $i = 1, \cdots, v-1$ and $j = 0, \cdots, v-1$. D is a $\mathsf{DCA}(v-1; 2, v, v)$.

We first construct an $\mathsf{OA}(v^3; v, v, 3)$ A by using Bush's construction (see the proof of Theorem 3.1 in [19]). The columns of A are labelled with the elements of $\mathbb{F}_v$ and rows are labelled by v^3 polynomials over $\mathbb{F}_v$ of degree at most 2. Then, in A, the entry in the column γ_i and the row labelled by the polynomial with coefficients β_0, β_1 and β_2 is $\beta_0 + \beta_1 \times \gamma_i + \beta_2 \times {\gamma_i}^2$.

Let B be the sub-array of A containing the rows of A which are labelled by the polynomials of degree 2 ($\beta_2 \neq 0$). Then B is a $(v^3 - v^2) \times v$ array. We label each column of B with the same element of $\mathbb{F}_v$ as its corresponding column in A. Denote i-th column of B by B_i, for $i = 0, \cdots, v-1$.

We produce a covering array $\mathsf{CA}(N'; 3, vk, v)$ G where $N' = N_3 + (v-1)N_2 + v^3 - v^2$. G is formed by vertically juxtaposing arrays G_1 of size $N_3 \times vk$, G_2 of size $(v-1)N_2 \times vk$, G_3 of size $(v^3 - v^2) \times vk$.

We describe the construction of each array in turn. We index vk columns by ordered pairs from $\{0, \ldots, k-1\} \times \{0, \ldots, v-1\}$.

G_1: In row r and column (f, h) place the entry in cell (r, f) of C_3. Thus G_1 consists of v copies of C_3 placed side by side.

G_2: Index the $(v-1)N_2$ rows by ordered pairs from $\{1,\ldots,N_2\}\times\{1,\ldots,v-1\}$. In row (r,s) and column (f,h) place $c_{r,f}+d_{s,h}$, where $c_{r,f}$ is the entry in cell (r,f) of C_2 and $d_{s,h}$ is the entry in cell (s,h) of D.

G_3: In row r and column (f,h) place the entry in cell (r,h) of B. Thus G_3 consists of k copies of B_0, the first column of B, then k copies of B_1, the second column, and so on.

We show that G is a 3-covering array. Consider three columns of G:

$$(f_1,h_1),(f_2,h_2),(f_3,h_3)$$

If f_1,f_2,f_3 are all distinct, then these columns restricted to G_1 arise from three distinct columns of C_3. Hence, all 3-tuples are covered.

If $f_1=f_2\neq f_3$ then all tuples of the form (x,x,y) are covered in G_1. All tuples of the form $(x+d_{y,h_1},x+d_{y,h_2},z+d_{y,h_3})$ for any $x,z\in\{0,1,\cdots,v-1\}$ and $y\in\{1,\cdots,v-1\}$ are covered in G_2. Therefore, since $h_1\neq h_2$ and D is a 2-difference covering array, it follows that all 3-tuples $(x,x+i,y)$ where $i\in\{1,\cdots,v\}$ and $x,y\in\{0,1,\cdots,v-1\}$ are covered in G_2.

If $f_1=f_2=f_3$ then $h_1\neq h_2\neq h_3$. All tuples of the form (x,x,x) are covered in G_1. All 3-tuples of the form $(x+d_{y,h_1},x+d_{y,h_2},x+d_{y,h_3})$, for any $x\in\{0,\cdots,v-1\}$ and $y\in\{1,\cdots,v-1\}$ are covered in G_2. Hence, for any $x,y\in\mathbb{F}_v$, all 3-tuples of the form $(x+y\times h_1,x+y\times h_2,x+y\times h_3)$ are covered in G_1 and G_2. The remaining 3-tuples of the form $(x+y\times h_1+z\times {h_1}^2,x+y\times h_2+z\times {h_2}^2,x+y\times h_3+z\times {h_3}^2)$, where $x,y\in\{0,\cdots,v-1\}$ and $z\in\{1,\cdots,v-1\}$, are covered in G_3. Hence all 3-tuples are covered. ■

4 Strength Four

In this section, we first establish general Roux-type constructions for strength four and then specialize them by restricting parameter values, and by employing specific ingredient arrays.

4.1 General Constructions

Theorem 4.1 *For max$(k,\ell)\geq 4$,*

$$\begin{aligned}\mathsf{CAN}(4,k\ell,v)\leq \mathsf{CAN}(4,k,v)+\mathsf{CAN}(4,\ell,v)+\mathsf{DCAN}(2,\ell,v)\mathsf{CAN}(3,k,v)\\ +\mathsf{DCAN}(2,k,v)\mathsf{CAN}(3,\ell,v)+\mathsf{QCAN}(k,\ell,v).\end{aligned}$$

Indeed when $k\geq 4$ and $\ell\geq 4$,

$$\begin{aligned}\mathsf{CAN}(4,k\ell,v)\leq \mathsf{CAN}(4,k,v)+\mathsf{CAN}(4,\ell,v)+\mathsf{DCAN}(2,\ell,v)\mathsf{CODN}(3,k,v)\\ +\mathsf{DCAN}(2,k,v)\mathsf{CODN}(3,\ell,v)+\mathsf{QCAN}(k,\ell,v).\end{aligned}$$

Proof. We prove the second statement, the first being a slight variation. Suppose that the following exist:

- $\mathsf{CA}(N_4;4,k,v)$ C_4,
- $\mathsf{CA}(R_4;4,\ell,v)$ B_4,

- $\mathsf{DCA}(S_1; 2, \ell, v)$ D_1,
- $\mathsf{COD}(N_3; 3, k, v)$ C_3,
- $\mathsf{DCA}(S_2; 2, k, v)$ D_2,
- $\mathsf{COD}(R_3; 3, \ell, v)$ B_3,
- $\mathsf{QCA}(M; k, \ell, v)$ G_5.

We produce a covering array $\mathsf{CA}(N'; 4, k\ell, v)$ G where $N' = N_4 + R_4 + N_3S_1 + R_3S_2 + M$. G is formed by vertically juxtaposing arrays G_1 of size $N_4 \times k\ell$, G_2 of size $R_4 \times k\ell$, G_3 of size $N_3S_1 \times k\ell$, G_4 of size $R_3S_2 \times k\ell$ and G_5 of size $M \times k\ell$. We describe the construction of G_1, G_2, G_3, and G_4 in turn. We index $k\ell$ columns by ordered pairs from $\{1, \ldots, k\} \times \{1, \ldots, \ell\}$.

G_1: In row r and column (f, h) place the entry in cell (r, f) of C_4. Thus G_1 consists of ℓ copies of C_4 placed side by side.

G_2: In row r and column (f, h) place the entry in cell (r, h) of B_4. Thus G_2 consists of k copies of the first column of B_4, then k copies of the second column, and so on.

G_3: Index the N_3S_1 rows by ordered pairs from $\{1, \ldots, N_3\} \times \{1, \ldots, S_1\}$. In row (r, s) and column (f, h) place $c_{r,f} + d_{s,h}$, where $c_{r,f}$ is the entry in cell (r, f) of C_3 and $d_{s,h}$ is the entry in cell (s, h) of D_1.

G_4: Index the S_2R_3 rows by ordered pairs from $\{1, \ldots, S_2\} \times \{1, \ldots, R_3\}$. In row (s, r) and column (f, h) place $b_{r,h} + d_{s,f}$, where $b_{r,h}$ is the entry in cell (r, h) of B_3 and $d_{s,f}$ is the entry in cell (s, f) of D_2.

We show that G is a 4-covering array. Consider four columns

$$(f_1, h_1), (f_2, h_2), (f_3, h_3), (f_4, h_4)$$

of G. If f_1, f_2, f_3, f_4 are all distinct, then these columns restricted to G_1 arise from four distinct columns of C_4. Hence, all 4-tuples are covered. Similarly, if h_1, h_2, h_3, h_4 are all distinct, then these four columns restricted to G_2 arise from distinct columns of B_4 and hence all 4-tuples are covered.

Further, we treat the following cases:

- $f_1 = f_2 \neq f_3 \neq f_4 \neq f_2$

  In this case $h_1 \neq h_2$. All 4-tuples (x, x, y, z) are covered in G_1, for any $x, y, z \in \{0, \cdots, v-1\}$.

  Now, suppose that $h_2 = h_3 = h_4$. Then G_3 covers all tuples of the form $(x, x+i, y+i, z+i)$ except where $x = y = z$: i.e. (x, w, w, w). These are exactly the tuples covered in G_2.

  Similarly, suppose that $h_1 = h_3 = h_4$. Then G_3 covers tuples of the form $(x, x+i, y, z)$ except for (x, w, x, x). These are covered in G_2.

  Suppose then that $h_1 = h_3$ and $h_2 = h_4$. G_3 covers tuples of the form $(x, x+i, y, z+i)$ except for $x = y = z$: i.e. (x, w, x, w). G_2 covers precisely tuples of this form. The argument is nearly identical if $h_1 = h_4$ and $h_2 = h_3$.

  Furthermore, suppose that $h_1 = h_3$, but $h_1 \neq h_2 \neq h_4 \neq h_1$. Then, G_3 covers tuples of the form $(x, x+i, y, z+j)$ except for $x = y = z$: i.e. (x, w, x, u). Again, G_2 covers all tuples of this form. Without loss of generality, cases with three distinct h values and $f_1 = f_2$ are treated in this manner.

Finally, assume that h_1, h_2, h_3, h_4 are distinct. This case has already been discussed. Hence all 4-tuples are covered for all possible sub-cases.

- $f_1 = f_2 = f_3 \neq f_4$

  In this case $h_1 \neq h_2 \neq h_3 \neq h_1$. The case where h_1, h_2, h_3 and h_4 are all distinct is discussed above. Suppose that $h_3 = h_4$, then 4-tuples (x, y, z, z) for any $x, y, z \in \{0, \cdots, v-1\}$ are covered in G_2. The 4-tuples $(x, y, z, z+i)$, for any $i \in \{1, \cdots, v-1\}$ and any $x, y, z \in \{0, \cdots, v-1\}$, are covered in G_4, except where $x = y = z$: i.e. (x, x, x, w). However, all tuples of this form are covered in G_1. Hence all 4-tuples are covered.

- $f_1 = f_2 \neq f_3 = f_4$

  In this case $h_1 \neq h_2$ and $h_3 \neq h_4$. Firstly, suppose that $h_2 = h_3$ but $h_1 \neq h_4$. Then 4-tuples (x, y, y, z) are covered in G_2 for any $x, y, z \in \{0, \cdots, v-1\}$. The 4-tuples $(x, y, y+i, z+i)$, for any $i, j \in \{1, \cdots, v-1\}$ and for any $x, y, z \in \{0, \cdots, v-1\}$, are covered in G_4 except where $x = y = z$: i.e. (x, x, w, w). These remaining tuples are covered in G_1. Hence all 4-tuples are covered.

  Now suppose that $h_2 = h_3$ and $h_1 = h_4$. Fix a 4-tuple (x, y, z, t) where x, y, z and t are any symbols from $\{0, \cdots, v-1\}$. If $x - t \equiv y - z \pmod{v}$, the 4-tuple is covering in $\mathsf{G}_1 - \mathsf{G}_4$; by the definition of the QCA, the remaining 4-tuples are covered by G_5.

■

Lemma 4.2 $\mathsf{QCAN}(k, \ell, v) \leq \mathsf{CODN}(2, k, \mathsf{CAN}(2, \ell, v))$.

Proof. Suppose that a $\mathsf{CA}(N; 2, \ell, v)$ C and a $\mathsf{COD}(R; 2, k, N)$ B both exist. A $\mathsf{QCA}(R; k, \ell, v)$ G is produced by replacing the symbol g in B by the gth row of C for all $g \in \{0, \ldots, N-1\}$. Columns of the resulting array are indexed by (i, j) where j indicates the column of B inflated, and i indexes the column of C within the row used in the inflation. Since C is a 2-covering array, it has a row i such that the entry in cell (i, f_1) is x and in cell (i, f_3) is t. C also contains a row j such that the entry in cell (j, f_1) is y and in the cell (j, f_3) is z. Furthermore, since B is a 2-COD on N symbols, it has a row m where the entry in cell (m, h_1) is the symbol i and in cell (m, h_2) is the symbol j. Thus, from the construction of G it follows that the tuple (x, y, z, t) with $x - t \not\equiv y - z \pmod{v}$ occurs in the row m and the columns $(f_1, h_1), (f_1, h_2), (f_3, h_2)$ and (f_3, h_1) of G. ■

Corollary 4.3 *For $k, \ell \geq 4$,*

$$\begin{aligned} \mathsf{CAN}(4, k\ell, v) \leq\ & \mathsf{CAN}(4, k, v) + \mathsf{CAN}(4, \ell, v) + \mathsf{DCAN}(2, \ell, v)\mathsf{CODN}(3, k, v) \\ & + \mathsf{DCAN}(2, k, v)\mathsf{CODN}(3, \ell, v) + \mathsf{CODN}(2, k, \mathsf{CAN}(2, \ell, v)). \end{aligned}$$

Proof. This follows from Theorem 4.1 and Lemma 4.2. ■

Lemma 4.4 $\mathsf{QCAN}(k, \ell, v) \leq \lceil \log_2 \ell \rceil \mathsf{QCAN}(k, 2, v)$.

Proof. Suppose that a $\mathsf{QCA}(N;k,2,v)$ C exists with columns indexed by $\{1\ldots,k\}\times\{0,1\}$. The $\mathsf{QCA}(k,\ell,v)$ G is constructed as follows. We index $k\ell$ columns by $\{1,\ldots,k\}\times\{1,\ldots,\ell\}$. Construct a binary array A with $\lceil\log_2\ell\rceil$ rows and ℓ distinct columns. For each row $(\rho_1,\ldots,\rho_\ell)$ of A in turn, form an $N\times k\ell$ array by replacing (in this row) the symbol $\rho_i\in\{0,1\}$ by the $N\times k$ subarray of C whose columns are indexed by $\{1,\ldots,k\}\times\{\rho_i\}$. Vertically juxtaposing the $\lceil\log_2\ell\rceil$ arrays so obtained produces G. ■

Lemma 4.5 $\mathsf{QCAN}(k,2,v)\leq\mathsf{CODN}(2,k,v^2)$.

Proof. Let C be a $\mathsf{COD}(N;2,k,v^2)$. Let ϕ be a one-to-one mapping from the symbols of C to $\{1,\ldots,v\}\times\{1,\ldots,v\}$. Construct two $N\times k$ arrays, E and F as follows. Let i be the entry in the cell (r,s) of C and $\phi(i)=(x,y)$. Then the entry in cell (r,s) of array E is x and the entry in cell (r,s) of array F is y. The QCA is produced by placing E and F side-by-side, indexing E by $\{1,\ldots,k\}\times\{1\}$ and F by $\{1,\ldots,k\}\times\{2\}$. ■

Corollary 4.6 *For $k,\ell\geq 4$,*

$$\mathsf{CAN}(4,k\ell,v)\leq\mathsf{CAN}(4,k,v)+\mathsf{CAN}(4,\ell,v)+\mathsf{DCAN}(2,\ell,v)\mathsf{CODN}(3,k,v)$$
$$+\mathsf{DCAN}(2,k,v)\mathsf{CODN}(3,\ell,v)+\lceil\log_2\ell\rceil\mathsf{CODN}(2,k,v^2).$$

Proof. This follows from Theorem 4.1 using Lemma 4.4 and Lemma 4.5. ■

4.2 Specializations when $\ell=2$

Hartman [17, 18] showed:

Theorem 4.7 $\mathsf{CAN}(4,2k,v)\leq\mathsf{CAN}(4,k,v)+(v-1)\mathsf{CAN}(3,k,v)+\mathsf{CAN}(2,k,v^2)$.

We derive a small improvement here.

Lemma 4.8 *For $k\geq 4$,*

$$\mathsf{CAN}(4,2k,v)\leq\mathsf{CAN}(4,k,v)+(v-1)\mathsf{CAN}(3,k,v)+\mathsf{CODN}(2,k,v)\mathsf{CODN}(2,v,v)+v\mathsf{CODN}(2,k,v)$$

Proof. Apply Theorem 4.1 with $\ell=2$, using Lemma 4.5 and Theorem 2.5. ■

Corollary 4.9 *For v a prime power and $k\geq 4$,*

$$\mathsf{CAN}(4,2k,v)\leq\mathsf{CAN}(4,k,v)+(v-1)\mathsf{CAN}(3,k,v)+v^2\mathsf{CAN}(2,k,v)-v^2$$

Proof. Use $\mathsf{CODN}(2,v,v)\leq v^2-v$ from Bush's orthogonal array construction, removing the v constant rows. Hence $\mathsf{CAN}(4,2k,v)\leq\mathsf{CAN}(4,k,v)+(v-1)\mathsf{CAN}(3,k,v)+v^2\mathsf{CODN}(2,k,v)$.

In addition, without loss of generality every $\mathsf{CA}(N;2,k,v)$ can have symbols renamed so that the resulting covering array has a constant row, whose deletion yields a $\mathsf{COD}(N-1;2,k,v)$. ■

4.3 Specializations when $v = 2$

We also provide a tripling specialization for binary arrays.

Theorem 4.10 $\mathsf{CAN}(4, 3k, 2) \leq \mathsf{CAN}(4, k, 2) + 6\mathsf{DCAN}(2, k, 2) + \mathsf{CAN}(3, k, 2) + \mathsf{CAN}(3, k+1, 2) + 4\mathsf{CODN}(2, k, 2)$

Proof. Suppose that the following exist:

- $\mathsf{CA}(N_4; 4, k, 2)$ C_4,
- $\mathsf{DCA}(S_2; 2, k, 2)$ D_2,
- $\mathsf{CA}(N_3; 3, k, 2)$ C_3,
- $\mathsf{CA}(M_3; 3, k+1, 2)$ F_3,
- $\mathsf{COD}(N_2; 2, k, 2)$ C_2.

Also, by removing the constant rows from Bush's orthogonal array, we can produce a

- $\mathsf{COD}(6; 3, 3, 2)$ B_3.

We produce a covering array $\mathsf{CA}(N'; 4, 3k, 2)$ G where $N' = N_4 + 6S_2 + N_3 + M_3 + 4N_2$. G is formed by vertically juxtaposing arrays G_1 of size $N_4 \times 3k$, G_4 of size $6S_2 \times 3k$, E_1 of size $N_3 \times 3k$, E_2 of size $M_3 \times 3k$, and K_1 through K_4 each of size $N_2 \times 3k$.

We describe the construction of each array in turn. We index $3k$ columns by ordered pairs from $\{0, ..., k-1\} \times \{0, 1, 2\}$.

The constructions of G_1 and G_4 are the same as those in Theorem 4.1. To produce the other ingredients, proceed as follows:

E_1: In row r and column $(f, 0)$ and $(f, 1)$ place the entry in cell (r, f) of C_3. In row r and column $(f, 2)$, place the bitwise complement of the entry in cell (r, f) of C_3.

E_2: Remove any column from F_3 to form a covering array of size $M_3 \times k$, F'_3 . In row r and column $(f, 0)$ place the entry in cell (r, f) of F'_3. In row r and column $(f, 1)$ place the bitwise complement of the entry in cell (r, f) of F'_3. In row r and column $(f, 2)$ place the r-th element of the column removed from F_3.

K_1: In row r and column $(f, 0)$ and $(f, 2)$ place the entry in cell (r, f) of C_2. In row r and column $(f, 1)$, place a 0.

K_2: In row r and column $(f, 1)$ and $(f, 2)$ place the entry in cell (r, f) of C_2. In row r and column $(f, 0)$, place a 0.

K_3: In row r and column $(f, 0)$ and $(f, 2)$ place the entry in cell (r, f) of C_2. In row r and column $(f, 1)$, place a 1.

K_4: In row r and column $(f, 1)$ and $(f, 2)$ place the entry in cell (r, f) of C_2. In row r and column $(f, 0)$, place a 1.

We show that G is a 4-covering array. Consider four columns

$$(f_1, h_1), (f_2, h_2), (f_3, h_3), (f_4, h_4)$$

of G. If f_1, f_2, f_3, f_4 are all distinct, then these columns restricted to G_1 arise from four distinct columns of C_4. Hence, all 4-tuples are covered. When $f_1 = f_2 = f_3 = f_4$, the values h_1, h_2, h_3 and h_4 must all be distinct, but this cannot occur as the h's are restricted to $\{0, 1, 2\}$.

Further, we need to consider the following cases:

- $f_1 = f_2 \neq f_3 \neq f_4 \neq f_2$

  In this case $h_1 \neq h_2$. Hence, the tuples (x, x, y, z) are covered in G_1. If no $h_i = 2$ then the tuples (x, x', y, z) for $x, y, z \in \{0, 1\}$ are covered in E_2. If h_1 or h_2 is 2, tuples (x, x', y, z) are covered in E_1.

  Without loss of generality, the remaining cases have $h_1 = 0$, $h_2 = 1$, $h_3 = 2$. Assume that $h_4 \neq 2$. Then the tuples (x, x', y, z) are covered in E_2. Finally, assume that $h_4 = 2$. Then, the tuples (x, x', y, y) are covered in E_2, leaving us to cover tuples of the form (x, x', y, y'). G_4 covers tuples of the form $(a + i, b + i, c, c')$ except for the case $a = b = c$, which is covered by G_1. Taking $a + i = x$, $b + i = x'$, and $c = y$, and hence $a \neq b$, we cover the remaining tuples in G_4.

- $f_1 = f_2 = f_3 \neq f_4$

  In this case $h_1 \neq h_2 \neq h_3 \neq h_1$. There are only three values for h_i, $i \in \{1, 2, 3, 4\}$; hence, without lost of generality, we suppose that $h_4 = h_1$.

  The tuples (x, x, x, y) are covered in G_1 for any $x, y \in \{0, 1\}$. The 4-tuples (x, y, z, x'), for any $x, y, z \in \{0, 1\}$ except $x = y = z$ are covered in G_4.

  This leaves six tuples: $(0, 0, 1, 0)$, $(1, 1, 0, 1)$, $(0, 1, 0, 0)$, $(1, 0, 0, 1)$, $(0, 1, 1, 0)$, and $(1, 0, 1, 1)$. We consider several cases for (h_1, h_2, h_3, h_4). When in one of these cases, all tuples are covered, any permutation of these indices also covers all tuples.

  If $h_1 = h_4 = 0$, $h_2 = 1$, and $h_3 = 2$, we cover tuples of the form (x, x, x', y) in E_1, treating $(0, 0, 1, 0)$ and $(1, 1, 0, 1)$. We cover tuples of the form (x, x', z, y) in E_2. This relies on the fact that F_3 can be split into two disjoint 2-covering arrays with k columns, one where the value in the column removed is 0 and one where the value in the column removed is 1. This treats the remaining cases.

  If $h_1 = h_4 = 1$, $h_2 = 0$, and $h_3 = 2$, we cover tuples of the form (x, x, x', y) in E_1, treating $(0, 0, 1, 0)$ and $(1, 1, 0, 1)$. We cover tuples of the form (x', x, z, y) in E_2. This eliminates the remaining cases.

  Finally, if $h_1 = h_4 = 2$, $h_2 = 0$ and $h_3 = 1$, we cover tuples of the form (x', x, x, y) in E_1, treating $(0, 1, 1, 0)$ and $(1, 0, 0, 1)$. We cover tuples of the form (x, y, y', x) in E_2, treating $(1, 1, 0, 1)$, $(1, 0, 1, 1)$, $(0, 0, 1, 0)$, and $(0, 1, 0, 0)$.

- $f_1 = f_2 \neq f_3 = f_4$

  In this case, $h_1 \neq h_2$ and $h_3 \neq h_4$. First, suppose that $h_2 = h_3$ but $h_1 \neq h_4$. Then 4-tuples (x, x, y, y) are covered in G_1. Tuples of the form (x, y, y', z') are covered in G_4, except when $x = y = z$, i.e. (x, x, x', x'). However these are exactly what G_1 covers. This leaves the six tuples of the form (x, y, y, z) with $x \neq z$ or $x \neq y$. We again consider specific cases for (h_1, h_2, h_3, h_4).

If $h_1 = 0$, $h_2 = h_3 = 1$, $h_4 = 2$, tuples of the form (x, x, y, y') are covered in E_1, which effectively covers tuples of the form (x, x, x, x'). In E_2, tuples of the form (x, x', y, z) are covered, which handles the remaining cases (x', x, x, z).

If $h_1 = 1$, $h_2 = h_3 = 0$, $h_4 = 2$, tuples of the form (x, x, y, y') are covered in E_1, which effectively covers tuples of the form (x, x, x, x'). In E_2, tuples of the form (x', x, y, z) are covered, which handles the remaining cases (x', x, x, z).

If $h_1 = 0$, $h_2 = h_3 = 2$, $h_4 = 1$, we cover tuples of the form (x, z, z, y) in E_2, which covers all required tuples.

Now suppose that $h_2 = h_3$ and $h_1 = h_4$. Tuples of the form (x, x, y, y) in G_1 and (x, y, y', x') are covered in G_4. The remaining tuples are $(0,1,1,0)$, $(1,0,0,1)$, $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$, $(0,1,1,1)$, $(1,0,1,1)$, $(1,1,0,1)$, and $(1,1,1,0)$.

If no $h_i = 2$, we cover (x, x', y, y') in E_2, treating $(0,1,1,0)$ and $(1,0,0,1)$, leaving us with all tuples comprised with an odd number of 0's. We cover $(x, 0, 0, x')$ and $(0, x, x', 0)$ in K_1 and K_2, and $(x, 1, 1, x')$ and $(1, x, x', 1)$ in K_3 and K_4. These are all the required cases.

Finally, without loss of generality, assume that $h_1 = h_4 = 2$. Then $h_2 = h_3 \in \{0, 1\}$. We cover (x, x', y, y') in E_1, again leaving us with the tuples having an odd number of 0's. We cover (x, y, z, x) in E_2. Here we again split F_3 into two 2-covering halves. This leaves only (x, y, y, x'), which are covered in K_2 and K_4 if $h_2 = 0$ or K_1 and K_3 if $h_2 = 1$.

Since all tuples are covered in all sets of four columns, G is the required covering array.

■

4.4 Specializations when $\ell = v = 3$

When $\ell = v = 3$ we have the following results:

Theorem 4.11

$$\mathsf{CAN}(4, 3k, 3) \leq \mathsf{CAN}(4, k, 3) + 2\mathsf{CAN}(3, k, 3) + 18\mathsf{DCAN}(2, k, 3) + \mathsf{CODN}(2, k, 9) + 18.$$

Proof. Suppose that the following exist:

- $\mathsf{CA}(N_4; 4, k, 3)$ C_4,
- $\mathsf{CA}(N_3; 3, k, 3)$ C_3,
- $\mathsf{DCA}(S; 2, k, 3)$ D,
- $\mathsf{CODN}(N_2; 2, k, 9)$ C_2,

Suppose that D' is the 2×3 array obtained by removing the first row from the $(3, 3; 1)$-difference matrix in Theorem 2.2. Then $d'_{i,j} = i \times j$ for $i = 1, 2$ and $j = 0, 1, 2$. The array D' is a $\mathsf{DCA}(2; 2, 3, 3)$.

Let A be an $\mathsf{OA}(27; 3, 3, 3)$ constructed by using Bush's construction.

The columns of A are labelled with the elements of $\mathbb{F}_3$ and rows are labelled by 27 polynomials over $\mathbb{F}_3$ of degree at most 2. Then the entry in A in the column labelled γ_i and the row labelled by the polynomial with coefficients β_0, β_1 and β_2 is $\beta_0 + \beta_1 \times \gamma_i + \beta_2 \times {\gamma_i}^2$.

Let A' be an $\mathsf{OA}(9; 2, 3, 3)$ which is also a $\mathsf{CA}(9; 2, 3, 3)$.

Let B be the sub-array of A containing the rows of A which are labelled by polynomials of degree 2 ($\beta_2 \neq 0$). Then B is a 18×3 array whose each column is labelled with the same element of $\mathbb{F}_3$ as its corresponding column in A. Denote the i-th column of B by B_i, for $i = 0, 1, 2$.

We produce a covering array $\mathsf{CA}(N'; 4, 3k, 3)$ G where $N' = N_4 + 2N_3 + 18S + N_2 + 18$. G is formed by vertically juxtaposing arrays G_1 of size $N_4 \times 3k$, G_2 of size $2N_3 \times 3k$, G_3 of size $18S \times 3k$, G_4 of size $N_2 \times 3k$ and G_5 of size $18 \times 3k$.

We describe the construction of each array in turn. We index $3k$ columns by ordered pairs from $\{0, \dots, k-1\} \times \{0, 1, 2\}$.

G_1: In row r and column (f, h) place the entry in cell (r, f) of C_4. Thus G_1 consists of three copies of C_4 placed side by side.

G_2: Index the $2N_3$ rows of G_2 by ordered pairs from $\{1, \dots, N_3\} \times \{1, 2\}$. In row (r, s) and column (f, h) place $c_{r,f} + d'_{s,h}$, where $c_{r,f}$ is the entry in cell (r, f) of C_3 and $d'_{s,h}$ is the entry in cell (s, h) of D'.

G_3: Index the $18S$ rows of G_3 by ordered pairs from $\{1, \dots, S\} \times \{1, \dots, 18\}$. In row (s, r) and column (f, h) place $b_{r,h} + d_{s,f}$, where $b_{r,h}$ is the entry in cell (r, h) of B and $d_{s,f}$ is the entry in cell (s, f) of D.

G_4: Define a mapping ϕ that maps the symbol i in C_2 to the 3-tuple in the i-th row of A', for $i \in \{0, \dots, 8\}$. Suppose that i is the symbol in cell (r, f) of C_2 and $\phi(i) = (x, y, z)$, for some $x, y, z \in \{0, 1, 2\}$. Then in row r and column $(f, 0)$ place the symbol x; in row r and column $(f, 1)$ place the symbol y; and in row r and column $(f, 2)$ place the symbol z.

G_5: In row r and column (f, h) place the entry in cell (r, h) of B. Thus G_5 consists of k copies of B_0, followed by k copies of B_1 and then k copies of B_2.

We show that G is a 4-covering array. Consider four columns

$$(f_1, h_1), (f_2, h_2), (f_3, h_3), (f_4, h_4)$$

of G. If f_1, f_2, f_3, f_4 are all distinct, then these columns restricted to G_1 arise from four distinct columns of C_4. Hence, all 4-tuples are covered. It cannot happen that $f_1 = f_2 = f_3 = f_4$ since then h_1, h_2, h_3 and h_4 are all distinct.

Further, we consider the following cases:

- $f_1 = f_2 \neq f_3 \neq f_4 \neq f_2$

  In this case $h_1 \neq h_2$. Hence, the tuples (x, x, y, z) are covered in G_1 and the tuples $(x, x{+}i, y, z)$ are covered in G_2 for any $x, y, z \in \{0, 1, 2\}$ and for any $i \in \{1, 2\}$.

- $f_1 = f_2 = f_3 \neq f_4$

  In this case $h_1 \neq h_2 \neq h_3 \neq h_1$. There are only 3 values for h_i, $i = 1, 2, 3, 4$, hence, without loss of generality, we suppose that $h_4 = h_1$.

The tuples (x, x, x, y) are covered in G_1 for any $x, y \in \{0, 1, 2\}$. The tuples $(x + d'_{y,h_1}, x + d'_{y,h_2}, x + d'_{y,h_3}, t + d'_{y,h_1})$ are covered in G_2 for any $x, t \in \{0, 1, 2\}$ and any $y \in \{1, 2\}$. Thus, all tuples $(x + yh_1, x + yh_2, x + yh_3, t)$ are covered in G_1 and in G_2 for any $x, y, t \in \{0, 1, 2\}$.

Further, the tuples $(x + yh_1 + zh_1^2, x + yh_2 + zh_2^2, x + yh_3 + zh_3^2, x + yh_1 + zh_1^2 + i)$, for any $x, y \in \{0, 1, 2\}$ and for $i, z \in \{1, 2\}$, are covered in G_3.

Finally, the tuples $(x+yh_1+zh_1^2, x+yh_2+zh_2^2, x+yh_3+zh_3^2, x+yh_1+zh_1^2)$, where $x, y \in \{0, 1, 2\}$ and $z \in \{1, 2\}$, are covered in G_5. Hence, all 4-tuples are covered.

- $f_1 = f_2 \neq f_3 = f_4$ In this case, $h_1 \neq h_2$ and $h_3 \neq h_4$. Firstly, suppose that $h_2 = h_3$ but $h_1 \neq h_4$.

  Fix any tuple (x, y, z, t) where $y \neq z$. Since A' is a 2-covering array, it has a row (x, y, m) for some $m \in \{0, 1, 2\}$, let it be i-th row. A' also has a row (s, z, t) for some $s \in \{0, 1, 2\}$, let it be j-th row. Since $y \neq z$ it follows that $i \neq j$. So $\phi(i) = (x, y, m)$ for the fixed x, y and for some m, and $\phi(j) = (s, z, t)$ for the fixed z, t and for some s. Since C_2 is a 2-COD and since $i \neq j$, C_2 has a row r such that in cell (r, f_1) is the symbol i and in cell (r, f_3) is the symbol j. Thus, the symbol x is in cell $(r, (f_1, h_1))$ of G_4, the symbol y is in cell $(r, (f_1, h_2))$ of G_4, the symbol z is in the cell $(r, (f_3, h_2))$ of G_4, and the symbol t is in the cell $(r, (f_3, h_4))$ of G_4. Hence, the fixed tuple (x, y, z, t) where $y \neq z$ is covered in G_4.

  Further, for $x \in \{0, 1, 2\}$, the tuple (x, x, x, x) is covered in G_1. The tuples $(x + y \times h_1, x + y \times h_2, x + y \times h_2, x + y \times h_4)$ are covered in G_2, for any $x \in \{0, 1, 2\}$ and any $y \in \{1, 2\}$. Tuples of the form $(x + y \times h_1 + z \times h_1^2, x + y \times h_2 + z \times h_2^2, x + y \times h_2 + z \times h_2^2, x + y \times h_4 + z \times h_4^2)$ are covered in G_5, for any $x, y \in \{0, 1, 2\}$ and any $z \in \{1, 2\}$. Hence all 4-tuples are covered.

  Now suppose that $h_2 = h_3$ and $h_1 = h_4$.

  Fix a tuple (x, y, z, t) such that if $x = t$ then $y \neq z$, for any $x, y, z, t \in \{0, 1, 2\}$. Since A' is a 2-covering array, it has a row (x, y, m) for some $m \in \{0, 1, 2\}$, let it be ith row. A' also has a row (t, z, s) for some $s \in \{0, 1, 2\}$, let it be jth row. Since $x \neq t$ or $y \neq z$ it follow that $i \neq j$. So $\phi(i) = (x, y, m)$ for the fixed x, y and for some m, and $\phi(j) = (t, z, s)$ for the fixed z, t and for some s. Since C_2 is a 2-COD and $i \neq j$, C_2 has a row r such that in cell (r, f_1) is the symbol i and in cell (r, f_3) is the symbol j. Thus, the symbol x is in cell $(r, (f_1, h_1))$ of G_4, the symbol y is in cell $(r, (f_1, h_2))$ of G_4, the symbol z is in the cell $(r, (f_3, h_2))$ of G_4, and the symbol t is in the cell $(r, (f_3, h_1))$ of G_4. Hence, the fixed tuple (x, y, z, t), where if $x = t$ then $y \neq z$, is covered.

  The tuples (x, x, x, x) are covered in G_1 for any $x \in \{0, 1, 2\}$. The tuples $(x + y \times h_1, x + y \times h_2, x + y \times h_2, x + y \times h_1)$ are covered in G_2 for any $x \in \{0, 1, 2\}$ and any $y \in \{1, 2\}$. So all tuples of the form (x, y, y, x) are covered in G_1 and in G_2.

■

Corollary 4.12

$$\mathsf{CAN}(4, 3k, 3) \leq \mathsf{CAN}(4, k, 3) + 2\mathsf{CAN}(3, k, 3) + 18\mathsf{DCAN}(2, k, 3) + \mathsf{CAN}(2, k, 9) - 1 + 18.$$

Proof. Without loss of generality every $\mathsf{CA}(N; 2, k, 9)$ can have symbols renamed so that the resulting covering array has a constant row, whose deletion yields a $\mathsf{COD}(N - 1; 2, k, 9)$. ■

4.5 Specializations when $\ell = v > 3$

Theorem 4.13 *For any prime power $v \geq 4$,*

$$\mathsf{CAN}(4, vk, v) \leq \mathsf{CAN}(4, k, v) + (v-1)\mathsf{CAN}(3, k, v) + (v^3 - v^2)\mathsf{DCAN}(2, k, v) + \mathsf{CODN}(2, k, v^2) + v^4 - v^2.$$

Proof. Suppose that the following exist:

- $\mathsf{CA}(N_4; 4, k, v)$ C_4,
- $\mathsf{CA}(N_3; 3, k, v)$ C_3,
- $\mathsf{DCA}(S; 2, k, v)$ D,
- $\mathsf{COD}(N_2; 2, k, v^2)$ C_2,

Suppose that D' is a $(v-1) \times v$ array obtained by removing the first row from the $(v, v; 1)$-difference matrix in Theorem 2.2. Then $d'_{i,j} = i \times j$ for $i = 1, \ldots, v-1$ and $j = 0, \ldots, v-1$. The array D' is a $\mathsf{DCA}(v-1; 2, v, v)$.

Let $\mathsf{A}^{(3)}$ be an $\mathsf{OA}(v^3; 3, v, v)$, constructed by using Bush's construction (see the proof of Theorem 3.1 in [19]). The columns of $\mathsf{A}^{(3)}$ are labelled with the elements of $\mathbb{F}_v$ and rows are labelled by v^3 polynomials over $\mathbb{F}_v$ of degree at most 2. Then, in $\mathsf{A}^{(3)}$, the entry in the column γ_i and the row labelled by the polynomial with coefficients β_0, β_1 and β_2 is $\beta_0 + \beta_1 \times \gamma_i + \beta_2 \times {\gamma_i}^2$.

Let $\mathsf{B}^{(3)}$ be the sub-array of $\mathsf{A}^{(3)}$ containing the rows of $\mathsf{A}^{(3)}$ which are labelled by polynomials of degree exactly 2 ($\beta_2 \neq 0$). Then $\mathsf{B}^{(3)}$ is a $(v^3 - v^2) \times v$ array. Label each column of $\mathsf{B}^{(3)}$ with the same element of $\mathbb{F}_v$ as its corresponding column in A. Denote the ith column of $\mathsf{B}^{(3)}$ by $\mathsf{B}_i^{(3)}$, for $i = 0, \ldots, v-1$.

Let $\mathsf{A}^{(4)}$ be an $\mathsf{OA}(v^4; 4, v, v)$ constructed by using Bush's construction. The columns of $\mathsf{A}^{(4)}$ are labelled with the elements of $\mathbb{F}_v$ and rows are labelled by v^4 polynomials over $\mathbb{F}_v$ of degree at most 3. Then, in $\mathsf{A}^{(4)}$, the entry in the column γ_i and the row labelled by the polynomial with coefficients $\beta_0, \beta_1, \beta_2$ and β_3 is $\beta_0 + \beta_1 \times \gamma_i + \beta_2 \times {\gamma_i}^2 + \beta_3 \times {\gamma_i}^3$.

Let $\mathsf{B}^{(4)}$ be the sub-array of $\mathsf{A}^{(4)}$ that contains the rows of $\mathsf{A}^{(4)}$ which are labelled by polynomials of degree 2 or 3($\beta_2 \neq 0$ or $\beta_3 \neq 0$). Then $\mathsf{B}^{(4)}$ is a $(v^4 - v^2) \times v$ array whose each column is labelled with the same element of $\mathbb{F}_v$ as its corresponding column in A. Denote the i-th column of $\mathsf{B}^{(4)}$ by $\mathsf{B}_i^{(4)}$, for $i = 0, \ldots, v-1$.

Let $\mathsf{A}^{(2)}$ be an $\mathsf{OA}(v^2; 2, v, v)$ which is also a $\mathsf{CA}(v^2; 2, v, v)$. Such an array exists by Theorem 2.1.

We produce a covering array $\mathsf{CA}(N'; 4, vk, v)$ G where $N' = N_4 + (v-1)N_3 + (v^3 - v^2)S + N_2 + v^4 - v^2$. G is formed by vertically juxtaposing arrays G_1 of size $N_4 \times vk$, G_2 of size $(v-1)N_3 \times vk$, G_3 of size $(v^3 - v^2)S \times vk$, G_4 of size $N_2 \times vk$ and G_5 of size $(v^4 - v^2) \times vk$.

We describe the construction of each array in turn. We index vk columns by ordered pairs from $\{0, \ldots, k-1\} \times \{0, \ldots, v-1\}$.

G_1: In row r and column (f, h) place the entry in cell (r, f) of C_4. Thus G_1 consists of v copies of C_4 placed side by side.

G_2: Index the $(v-1)N_3$ rows by ordered pairs from $\{1,\dots,N_3\}\times\{1,\dots,v-1\}$. In row (r,s) and column (f,h) place $c_{r,f}+d'_{s,h}$, where $c_{r,f}$ is the entry in cell (r,f) of C_3 and $d'_{s,h}$ is the entry in cell (s,h) of D'.

G_3: Index the $(v^3-v^2)S$ rows by ordered pairs from $\{1,\dots,S\}\times\{1,\dots,(v^3-v^2)\}$. In row (s,r) and column (f,h) place $b_{r,h}+d_{s,f}$, where $b_{r,h}$ is the entry in cell (r,h) of $\mathsf{B}^{(3)}$ and $d_{s,f}$ is the entry in cell (s,f) of D.

G_4: Let ϕ be a mapping that maps the symbol i of C_2 to the v-tuple on the i-th row of $\mathsf{A}^{(2)}$, for any $i=\{0,\dots,v^2-1\}$. Let i be the symbol in cell (r,f) in C_2. Suppose that $\phi(i)=(x_0,x_1,\dots,x_{v-1})$ for some $x_0,x_1,\dots,x_{v-1}\in\mathbb{F}_v$. Then, in row r and column (f,m) place the symbol x_m, for $m=0,\dots,v-1$.

G_5: In row r and column (f,h) place the entry in cell (r,h) of $\mathsf{B}^{(4)}$. Thus G_5 consists of k copies of the first column of $\mathsf{B}^{(4)}$, followed by k copies of the second column of $\mathsf{B}^{(4)}$, and so on.

We show that G is a 4-covering array. Consider four columns

$$(f_1,h_1),(f_2,h_2),(f_3,h_3),(f_4,h_4)$$

of G. If f_1,f_2,f_3,f_4 are all distinct, then these columns restricted to G_1 arise from four distinct columns of C_4. Hence, all 4-tuples are covered.

Further, we consider the following cases:

- $f_1=f_2\neq f_3\neq f_4\neq f_2$

  All 4-tuples (x,x,y,z) are covered in G_1, for any $x,y,z\in\{0,\cdots,v-1\}$. All 4-tuples $(x,x+i,y,z)$, for any $i\in\{1,\cdots,v-1\}$ and any $x,y,z\in\{0,\cdots,v-1\}$, are covered in G_2. Hence all 4-tuples are covered.

- $f_1=f_2=f_3\neq f_4$

  In this case $h_1\neq h_2\neq h_3\neq h_1$. The case where h_1, h_2, h_3 and h_4 are all distinct is discussed separately. Now suppose that $h_4=h_1$.

  The tuples (x,x,x,y), for any $x,y\in\{0,\cdots,v-1\}$, are covered in G_1. The tuples $(x+d'_{y,h_1},x+d'_{y,h_2},x+d'_{y,h_3},t+d'_{y,h_1})$, for any $x,t\in\{0,\cdots,v-1\}$ and for $y\in\{1,\cdots,v-1\}$, are covered in G_2.

  So all the tuples $(x+yh_1,x+yh_2,x+yh_3,t)$, for any $x,y,t\in\{0,\dots,v-1\}$, are covered in G_1 and in G_2.

  The tuples $(x+yh_1+zh_1^2,x+yh_2+zh_2^2,x+yh_3+zh_3^2,x+yh_1+zh_1^2+i)$, where $i,z\in\{1,\cdots,v-1\}$ and $x,y\in\{0,\cdots,v-1\}$, are covered in G_3. Finally, the tuples $(x+yh_1+zh_1^2+th_1^3,x+yh_2+zh_2^2+th_2^3,x+yh_3+zh_3^2+th_3^3,x+yh_1+zh_1^2+th_1^3,)$, where if $z=0$ then $t\neq 0$ for any $x,y,z,t\in\{0,\dots,v-1\}$, is covered in G_5. Hence, all 4-tuples are covered.

- $f_1=f_2\neq f_3=f_4$ and $h_2=h_3$ but $h_1\neq h_4$.

  In this case $h_1\neq h_2$ and $h_3\neq h_4$.

  Fix any tuple (x,y,z,t) where $y\neq z$. Since $\mathsf{A}^{(2)}$ is a 2-covering array, it has row with the tuple $(m_0,\dots,m_{v-1})$, where $m_{h_1}=x$ and $m_{h_2}=y$, let it be ith row of $\mathsf{A}^{(2)}$. $\mathsf{A}^{(2)}$ also has a row with the tuple $(m'_0, \ldots, m'_{v-1})$, where $m'_{h_2} = z$ and $m'_{h_4} = t$, let it be row jth row of $\mathsf{A}^{(2)}$. Since $y \neq z$ it follows that $i \neq j$. So $\phi(i) = (m_0, \ldots, m_{v-1})$ and $\phi(j) = (m'_0, \ldots, m'_{v-1})$. Since C_2 is a 2-COD and $i \neq j$, C_2 has a row r such that in cell (r, f_1) is the symbol i and in cell (r, f_3) is the symbol j. Thus, in G_4, the symbol x is in cell $(r, (f_1, h_1))$, the symbol y is in cell $(r, (f_1, h_2))$, the symbol z is in cell $(r, (f_3, h_2))$ and the symbol t is in cell $(r, (f_3, h_4))$. Hence, the fixed tuple (x, y, z, t) is covered when $y \neq z$.

Further, the tuple (x, x, x, x), for any $x \in \{0, \ldots, v-1\}$, is covered in G_1. The tuple $(x + yh_1, x + yh_2, x + yh_2, x + yh_4)$, for any $x \in \{0, \ldots, v-1\}$ and any $y \in \{1, \ldots, v-1\}$, is covered in G_2.

Finally, the tuples $(x+yh_1+zh_1^2+th_1^3, x+yh_2+zh_2^2+th_2^3, x+yh_2+zh_2^2+th_2^3, x+yh_4+zh_4^2+th_4^3)$, such that if $z = 0$ then $t \neq 0$, for any $x, y, z, t \in \{0, \ldots, v-1\}$, are covered in G_5.

- $f_1 = f_2 \neq f_3 = f_4$, $h_2 = h_3$ and $h_1 = h_4$.

  Fix any tuple (x, y, z, t) such that if $x = t$ then $y \neq z$. Since $\mathsf{A}^{(2)}$ is a 2-covering array, it has row with the tuple $(m_0, \ldots, m_{v-1})$, where $m_{h_1} = x$ and $m_{h_2} = y$, let it be ith row of $\mathsf{A}^{(2)}$. $\mathsf{A}^{(2)}$ also has a row with the tuple $(m'_0, \ldots, m'_{v-1})$, where $m'_{h_1} = t$ and $m'_{h_2} = z$, let it be jth row $\mathsf{A}^{(2)}$. Since either $x \neq t$ or $y \neq z$ it follows that $i \neq j$. Now $\phi(i) = (m_0, \ldots, m_{v-1})$ and $\phi(j) = (m'_0, \ldots, m'_{v-1})$.

  Since C_2 is a 2-COD and $i \neq j$, it has a row r such that in cell (r, f_1) is the symbol i and in cell (r, f_3) is the symbol j. Thus, in G_4, the symbol x is in cell $(r, (f_1, h_1))$ the symbol y is in cell $(r, (f_1, h_2))$ the symbol z is in the cell $(r, (f_3, h_2))$ and the symbol t is in the cell $(r, (f_3, h_1))$. Hence, any fixed tuple (x, y, z, t), such that if $x = t$ then $y \neq z$, for any $x, y, z, t \in \{0, \ldots, v-1\}$, is covered in G_4.

  Further, the tuples of the form (x, x, x, x) are covered in G_1. The tuples of the form $(x + yh_1, x + yh_2, x + yh_2, x + yh_1)$ are covered in G_2 for $x \in \{0, \ldots, v-1\}$ and $y \in \{1, \ldots, v-1\}$.

  These are all the tuples of the form (x, y, y, x) for any $x, y \in \{0, \ldots, v-1\}$. Hence all 4-tuples are covered.

- In the remaining cases which are not discussed above h_1, h_2, h_3 and h_4 are all distinct.

  The tuple (x, x, x, x) is covered in G_1 for any $x \in \{0, \ldots, v-1\}$. The tuple

  $(x + yh_1, x + yh_2, x + yh_3, x + yh_4)$ is covered in G_2 for any $x \in \{0, \ldots, v-1\}$ and any $y \in \{1, \ldots, v-1\}$. Finally, the tuple $(x + yh_1 + zh_1^2 + th_1^3, x + yh_2 + zh_2^2 + th_2^3, x + yh_3 + zh_3^2 + th_3^3, x + yh_4 + zh_4^2 + th_4^3)$ such that if $z = 0$ then $t \neq 0$, for any $x, y, z, t \in \{0, \ldots, v-1\}$, is covered in G_5.

■

Corollary 4.14 *For any prime power $v \geq 4$,*

$$\begin{aligned} \mathsf{CAN}(4, vk, v) \leq\; & \mathsf{CAN}(4, k, v) + (v-1)\mathsf{CAN}(3, k, v) + \\ & (v^3 - v^2)\mathsf{DCAN}(2, k, v) + \mathsf{CAN}(2, k, v^2) - 1 + v^4 - v^2. \end{aligned}$$

Proof. Without loss of generality every $\mathsf{CA}(N; 2, k, v^2)$ can have symbols renamed so that the resulting covering array has a constant row, whose deletion yields a $\mathsf{COD}(N-1; 2, k, v^2)$. ■

Corollary 4.15 *For any prime power $v \geq 4$,*

$$\begin{aligned}\mathsf{CAN}(4, vk, v) \quad &\leq \quad \mathsf{CAN}(4, k, v) + (v-1)\mathsf{CAN}(3, k, v) + \\ &\quad\;\; (v^3 - v^2)\mathsf{DCAN}(2, k, v) + (v^2 + v)\mathsf{CAN}(2, k, v) - 1 + v^4 - 2v^2.\end{aligned}$$

Proof. Apply Corollary 2.4 to bound $\mathsf{CAN}(2, k, v^2)$. ■

5 Numerical Consequences

To assess the effectiveness of the recursions developed, it is necessary to determine their impact on our knowledge of covering array numbers. We have outlined computational methods in the introduction; in preparation for a comparison we therefore introduce related recursive methods that do not (at present) fall into the "Roux-type" framework.

The *Turán number* $T(t, n)$ is the largest number of edges in a t-vertex simple graph having no $(n+1)$-clique. Turán [32] showed that a graph with the $T(t, n)$ edges is constructed by setting $a = \lfloor t/n \rfloor$ and $b = t - na$, and forming a complete multipartite graph with b classes of size $a+1$ and $n - b$ classes of size a. Using these, Hartman generalizes a constructions in [6, 7, 30].

Theorem 5.1 *[17] If a $\mathsf{CA}(N; t, k, v)$ and a $\mathsf{CA}(k^2; 2, T(t, v) + 1, k)$ both exist, then a $\mathsf{CA}(N \cdot (T(t, v) + 1); t, k^2, v)$ exists.*

Perfect hash families are well studied combinatorial objects. A *t-perfect hash family $\mathcal{H}$*, denoted $\mathsf{PHF}(n; k, q, t)$, is a family of n functions $h : A \mapsto B$, where $k = |A| \geq |B| = q$, such that for any subset $X \subseteq A$ with $|X| = t$, there is at least one function $h \in \mathcal{H}$ that is injective on X. Thus a $\mathsf{PHF}(n; k, q, t)$ can be viewed as an $n \times k$-array $\mathcal{H}$ with entries from a set of q symbols such that for any set of t columns there is at least one row having distinct entries in this set of columns.

Theorem 5.2 *(see [3, 23]) If a $\mathsf{PHF}(s; k, m, t)$ and a $\mathsf{CA}(N; t, m, v)$ both exist then a $\mathsf{CA}(sN; t, k, v)$ exists.*

For constructions of perfect hash families, see [1, 2, 4, 5, 31].

To assess the contributions of each of the constructions described, we computed upper bounds for $\mathsf{CAN}(t, k, v)$ for $t \in \{2, 3, 4\}$, $2 \leq v \leq 25$, and $t < k \leq 10000$. Previous tables (e.g., [8]) have reported only small numbers of factors ($k \leq 30$). With the current power of computational search techniques, this fails to explore into the range in which recursions are most powerful. Evidently it is not sensible to report 10,000 results for every t and v, and fortunately there is no need to do so. Let $\kappa(N; t, v)$ be the largest k for which $\mathsf{CAN}(t, k, v) \leq N$. As k increases, for many consecutive numbers of factors, the covering array number does not change. Therefore reporting those values of $\kappa(N; t, v)$ for which $\kappa(N; t, v) > \kappa(N-1; t, v)$, along with the corresponding value of N, enables one to determine all covering array numbers when k is no larger than the largest $\kappa(N; t, v)$ value tabulated. Since the exact values for covering array numbers are unknown in general, we in fact report lower bounds on $\kappa(N; t, v)$.

For each strength in turn, explicit constructions of covering arrays from direct and computational constructions are tabulated. Then each known construction is applied and its consequences tabulated (in the process, results implied by this for fewer factors are suppressed, so that one explanation ("authority") for each entry is maintained). Applications of the recursions is repeated until no entries in the table improve.

The authorities used are:

f	constraint programming [20]	h	perfect hash family [23]
ℓ	Roux-type [10]	m	Roux-type (this paper)
n	nearly resolvable design [8]	o	orthogonal array [19]
q	Turán squaring [17]	r	Roux-type (this paper)
s	simulated annealing [9]	t	tabu search [25]
u	Martirosyan (unpublished)	v	permutation vector [33]
y	binary construction [28]	z	composition
$\downarrow$	symbol identification		

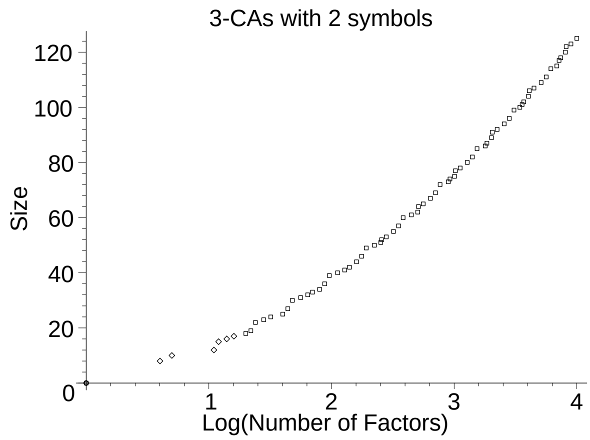
Composition and symbol identification are standard constructions; see [8], for example. Other constructions, such as derivation of a t-covering array from a $(t+1)$-covering array, and "Construction D" from [8], can yield improvements but do not do so within the ranges of the tables reported; hence they are omitted.

5.1 Tables for Strength Three

We provide tables for (lower bounds on) $\kappa(N;3,v)$ for $2 \leq v \leq 9$ only, since they illustrate the main points. The strength two tables used are from [13]. For each v, we tabulate the entries for N and $\kappa(N;3,v)$. We also provide a plot showing the logarithm of the number of factors horizontally and the size of the covering array vertically. Asymptotically one expects this to become a straight line (see, e.g., [16]), and its deviation from the straight line results from non-uniform behaviour when k is small, but also from the "errors" compounded in repeated applications of the recursions. The plot simply demonstrates the growth; the explicit points given are definitive.

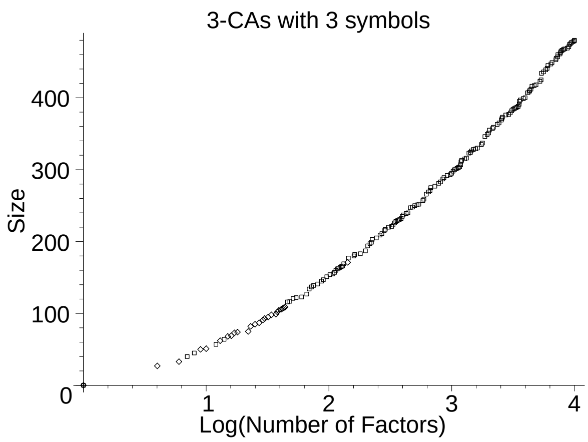
Exponents indicate the authority for the entry provided, to provide one method for the construction; alternative constructions may produce the same result.

4	8^{o}	5	10^{n}	11	12^{y}
12	15^{t}	14	16^{y}	16	17^{y}
20	18^{ℓ}	22	19^{ℓ}	24	22^{ℓ}
28	23^{ℓ}	32	24^{ℓ}	40	25^{ℓ}
44	27^{ℓ}	48	30^{ℓ}	56	31^{ℓ}
64	32^{ℓ}	70	33^{ℓ}	80	34^{ℓ}
88	36^{ℓ}	96	39^{ℓ}	112	40^{ℓ}
128	41^{ℓ}	140	42^{ℓ}	160	44^{ℓ}
176	46^{ℓ}	192	49^{ℓ}	224	50^{ℓ}
252	51^{ℓ}	256	52^{ℓ}	280	53^{ℓ}
320	55^{ℓ}	352	57^{ℓ}	384	60^{ℓ}
448	61^{ℓ}	504	62^{ℓ}	512	64^{ℓ}



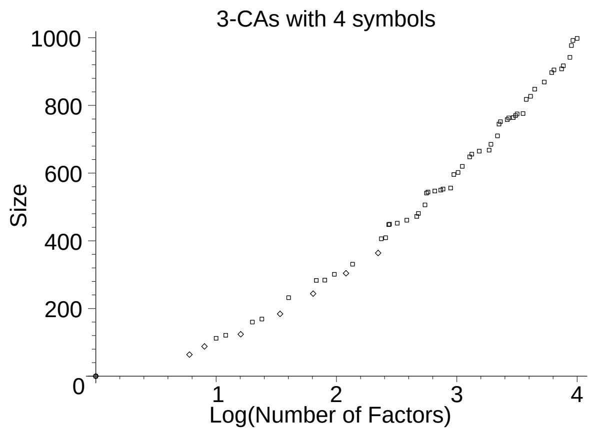
560	65^ℓ	640	67^ℓ	704	69^ℓ	768	72^ℓ	896	73^ℓ	924	74^ℓ
1008	75^ℓ	1024	77^ℓ	1120	78^ℓ	1280	80^ℓ	1408	82^ℓ	1536	85^ℓ
1792	86^ℓ	1848	87^ℓ	2016	89^ℓ	2048	91^ℓ	2240	92^ℓ	2560	94^ℓ
2816	96^ℓ	3072	99^ℓ	3432	100^ℓ	3584	101^ℓ	3696	102^ℓ	4032	104^ℓ
4096	106^ℓ	4480	107^ℓ	5120	109^ℓ	5632	111^ℓ	6144	114^ℓ	6864	115^ℓ
7168	117^ℓ	7392	118^ℓ	8064	120^ℓ	8192	122^ℓ	8960	123^ℓ	10000	125^ℓ

4	27^o	6	33^n	7	40^f
8	45^ℓ	9	50^s	10	51^v
12	57^ℓ	13	62^s	14	64^ℓ
15	68^s	16	69^s	17	73^s
18	74^s	22	75^v	23	82^s
25	85^s	27	87^s	29	91^s
30	93^s	32	95^s	34	98^s
37	99^v	38	102^s	39	104^s
40	105^ℓ	41	106^s	42	107^s
43	108^s	44	109^s	46	116^ℓ
48	117^m	51	121^m	54	122^m
60	123^m	66	127^m	69	134^m



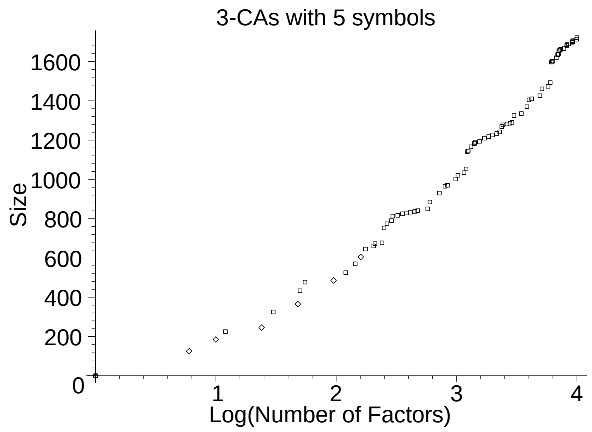
72	137^ℓ	75	139^m	81	141^m	87	145^m	90	147^m	96	151^m
102	154^m	108	155^m	111	157^m	114	160^m	117	162^m	120	163^m
123	164^m	126	165^m	129	166^m	132	169^m	142	171^v	144	177^m
160	180^ℓ	162	182^m	180	183^m	198	187^m	207	194^m	216	197^m
222	199^m	225	203^m	243	205^m	261	209^m	270	211^m	282	215^m
288	217^m	306	220^m	324	221^m	333	223^m	342	226^m	351	228^m
360	229^m	369	230^m	378	231^m	387	232^m	396	235^m	402	237^m
426	239^m	440	240^ℓ	460	247^ℓ	480	248^m	500	250^ℓ	522	251^m
540	252^ℓ	582	257^m	594	259^m	621	266^m	648	269^m	666	271^m
675	275^m	729	277^m	783	281^m	810	283^m	846	287^m	864	289^m
918	292^m	972	293^m	999	295^m	1026	298^m	1053	300^m	1080	301^m
1107	302^m	1134	303^m	1161	304^m	1182	307^m	1188	311^m	1206	313^m
1278	315^m	1320	316^m	1380	323^m	1422	324^m	1440	326^m	1500	328^m
1566	329^m	1620	330^m	1746	335^m	1782	337^m	1863	346^m	1944	349^m
1998	351^m	2025	355^m	2142	357^m	2187	359^m	2349	363^m	2430	365^m
2538	369^m	2562	371^m	2592	373^m	2754	376^m	2916	377^m	2997	379^m
3078	382^m	3159	384^m	3240	385^m	3321	386^m	3402	387^m	3483	388^m
3546	391^m	3564	395^m	3618	397^m	3834	399^m	3960	400^m	4140	407^m
4266	408^m	4320	410^m	4422	412^m	4500	416^m	4698	417^m	4860	418^m
5238	423^m	5346	425^m	5388	434^m	5589	436^m	5832	439^m	5994	441^m
6075	445^m	6426	447^m	6561	449^m	7047	453^m	7092	455^m	7290	457^m
7326	460^ℓ	7614	461^m	7686	463^m	7776	465^m	7920	466^ℓ	8118	467^ℓ
8316	468^ℓ	8748	469^m	8991	471^m	9090	474^m	9234	475^ℓ	9477	477^ℓ
9720	478^ℓ	9963	479^ℓ	10000	480^ℓ						

6	64^{o}	8	88^{n}	10	112^{ℓ}
12	121^{ℓ}	16	124^{v}	20	160^{m}
24	169^{m}	34	184^{v}	40	232^{m}
64	244^{v}	68	283^{ℓ}	80	284^{ℓ}
96	301^{m}	120	304^{v}	136	331^{m}
222	364^{v}	236	406^{m}	256	409^{m}
272	448^{m}	276	449^{m}	320	452^{ℓ}
384	461^{ℓ}	464	472^{m}	480	481^{m}
544	506^{ℓ}	560	541^{m}	576	544^{m}
656	547^{m}	736	550^{m}	768	553^{m}
888	556^{m}	944	596^{ℓ}	1024	602^{ℓ}
1110	620^{ℓ}	1280	648^{ℓ}	1332	656^{ℓ}



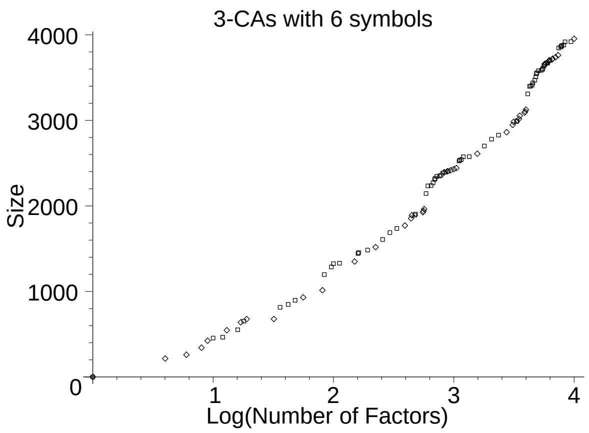
1536	665^{ℓ}	1856	668^{ℓ}	1920	685^{m}	2176	710^{m}	2240	745^{m}	2304	752^{ℓ}
2624	758^{ℓ}	2704	763^{m}	2944	764^{ℓ}	3072	770^{ℓ}	3168	775^{m}	3552	776^{ℓ}
3776	818^{ℓ}	4096	827^{ℓ}	4440	848^{m}	5328	869^{ℓ}	6144	897^{ℓ}	6416	905^{m}
7424	908^{ℓ}	7680	917^{ℓ}	8704	942^{ℓ}	8960	977^{ℓ}	9216	992^{m}	10000	998^{m}

6	125^{o}	10	185^{n}	12	225^{ℓ}
24	245^{v}	30	325^{m}	48	365^{v}
50	433^{m}	55	477^{m}	95	485^{v}
120	525^{m}	144	570^{ℓ}	160	605^{v}
175	645^{m}	205	661^{m}	210	673^{m}
240	677^{m}	250	753^{m}	264	774^{ℓ}
288	790^{ℓ}	295	813^{m}	325	817^{m}
355	825^{m}	385	829^{m}	415	833^{m}
450	837^{m}	475	841^{m}	576	850^{ℓ}
600	885^{m}	720	930^{m}	800	965^{m}
840	970^{ℓ}	984	1002^{ℓ}	1025	1021^{m}
1152	1034^{ℓ}	1200	1053^{m}	1225	1141^{m}



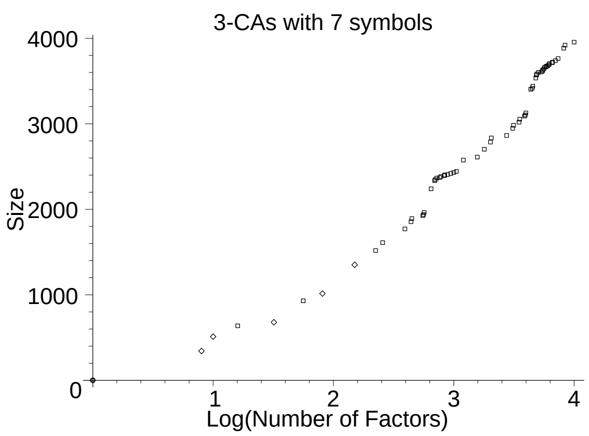
1250	1145^{m}	1320	1166^{m}	1405	1182^{m}	1416	1186^{ℓ}	1440	1190^{m}	1560	1194^{ℓ}
1704	1210^{ℓ}	1848	1218^{ℓ}	1992	1226^{ℓ}	2160	1234^{ℓ}	2280	1242^{ℓ}	2375	1269^{m}
2425	1278^{m}	2625	1282^{m}	2775	1286^{m}	2880	1290^{m}	3000	1325^{m}	3456	1335^{ℓ}
3840	1370^{ℓ}	4000	1405^{m}	4200	1410^{m}	4920	1426^{ℓ}	5125	1461^{m}	5760	1474^{m}
6000	1493^{m}	6125	1597^{m}	6250	1601^{m}	6336	1603^{ℓ}	6744	1619^{ℓ}	6912	1635^{ℓ}
7025	1638^{m}	7080	1654^{m}	7175	1658^{m}	7320	1662^{ℓ}	7800	1666^{m}	8225	1682^{m}
8280	1686^{ℓ}	8520	1690^{m}	9120	1698^{ℓ}	9225	1702^{m}	9240	1706^{m}	9960	1714^{m}
10000	1722^{m}										

4	216^{o}	6	260^{s}	8	$342^{\downarrow}$
9	423^{s}	10	455^{ℓ}	12	465^{ℓ}
13	546^{s}	16	552^{ℓ}	17	638^{s}
18	653^{ℓ}	19	677^{s}	32	$678^{\downarrow}$
36	814^{ℓ}	42	848^{m}	48	896^{ℓ}
56	$930^{\downarrow}$	81	$1014^{\downarrow}$	84	1197^{m}
96	1286^{ℓ}	100	1325^{ℓ}	112	1330^{ℓ}
150	$1350^{\downarrow}$	160	1444^{ℓ}	162	1454^{ℓ}
192	1484^{ℓ}	224	$1518^{\downarrow}$	256	1608^{ℓ}
294	1688^{m}	336	1736^{m}	392	$1770^{\downarrow}$
441	$1854^{\downarrow}$	448	$1890^{\downarrow}$	474	1892^{ℓ}
480	1904^{ℓ}	553	$1926^{\downarrow}$	560	$1938^{\downarrow}$



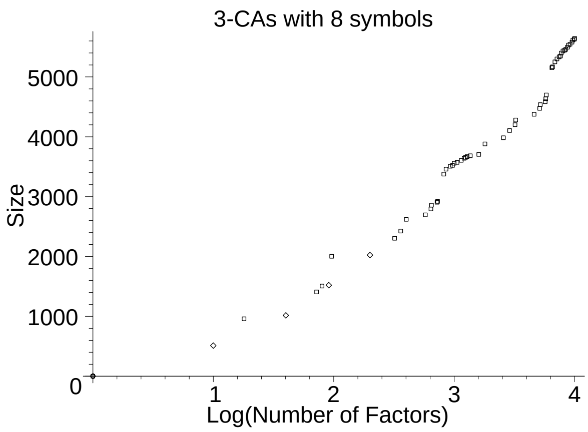
567	$1962^{\downarrow}$	588	2145^{m}	609	2234^{m}	648	2238^{ℓ}	672	2270^{m}	693	2309^{m}
700	2321^{m}	721	2344^{m}	763	2350^{m}	784	2360^{ℓ}	810	2384^{ℓ}	833	$2394^{\downarrow}$
858	2396^{ℓ}	889	$2406^{\downarrow}$	900	2408^{ℓ}	945	$2418^{\downarrow}$	1001	$2430^{\downarrow}$	1050	$2442^{\downarrow}$
1106	2526^{ℓ}	1120	2536^{m}	1152	2542^{ℓ}	1200	2574^{ℓ}	1344	2576^{m}	1568	$2610^{\downarrow}$
1792	2700^{m}	2058	2780^{m}	2352	2828^{m}	2744	$2862^{\downarrow}$	3087	$2946^{\downarrow}$	3136	$2982^{\downarrow}$
3318	2984^{m}	3360	2996^{m}	3479	$3018^{\downarrow}$	3528	$3054^{\downarrow}$	3871	$3090^{\downarrow}$	3920	$3102^{\downarrow}$
3969	$3126^{\downarrow}$	4116	3309^{m}	4263	3398^{m}	4361	3402^{m}	4480	3414^{m}	4536	3438^{m}
4704	3470^{m}	4802	3509^{m}	4851	3545^{m}	4900	3557^{m}	5047	3580^{m}	5341	3586^{m}
5467	3596^{m}	5488	3608^{m}	5600	3632^{m}	5670	3650^{m}	5684	$3660^{\downarrow}$	5831	$3666^{\downarrow}$
6006	3668^{m}	6020	$3678^{\downarrow}$	6174	$3690^{\downarrow}$	6223	$3702^{\downarrow}$	6300	3704^{m}	6566	$3714^{\downarrow}$
6615	$3720^{\downarrow}$	7007	$3738^{\downarrow}$	7350	$3762^{\downarrow}$	7448	3846^{m}	7742	3858^{m}	7840	3868^{m}
7889	3874^{m}	8192	3882^{ℓ}	8400	3918^{m}	9408	3920^{m}	10000	$3954^{\downarrow}$		

8	343^{o}	10	$511^{\downarrow}$	16	637^{ℓ}
32	679^{v}	56	931^{m}	81	1015^{v}
150	1351^{v}	224	1519^{m}	256	1610^{ℓ}
392	1771^{m}	441	1855^{m}	448	1891^{m}
553	1927^{m}	560	1939^{m}	567	1963^{m}
648	2240^{ℓ}	693	2335^{m}	700	2347^{m}
721	2365^{m}	763	2371^{m}	784	2383^{m}
833	2395^{m}	840	2401^{m}	889	2407^{m}
945	2419^{m}	1001	2431^{m}	1050	2443^{m}
1200	2576^{ℓ}	1568	2611^{m}	1792	2702^{m}
2016	2786^{ℓ}	2048	2835^{ℓ}	2744	2863^{m}
3087	2947^{m}	3136	2983^{m}	3479	3019^{m}



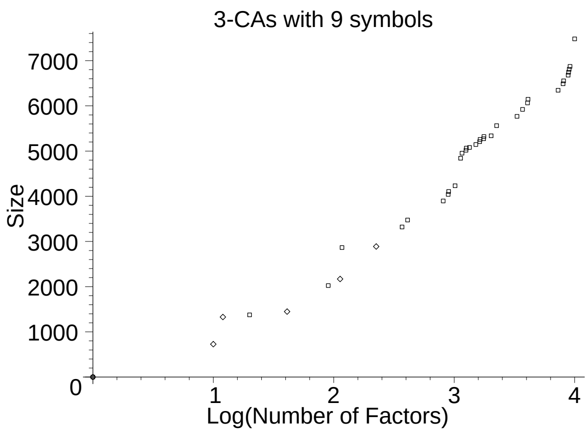
3528	3055^{m}	3871	3091^{m}	3920	3103^{m}	3969	3127^{m}	4361	3404^{m}	4480	3416^{m}
4536	3440^{m}	4802	3535^{m}	4851	3571^{m}	4900	3583^{m}	5047	3601^{m}	5341	3607^{m}
5467	3619^{m}	5488	3631^{m}	5600	3643^{m}	5684	3661^{m}	5831	3667^{m}	5880	3673^{m}
6020	3679^{m}	6174	3691^{m}	6223	3703^{m}	6566	3715^{m}	6615	3721^{m}	7007	3739^{m}
7350	3763^{m}	8192	3885^{ℓ}	8400	3920^{m}	10000	3955^{m}				

10	512^o	18	960^ℓ	40	1016^v
72	1408^m	80	1506^m	91	1520^v
96	2003^m	200	2024^v	320	2304^m
360	2424^ℓ	400	2620^ℓ	576	2696^m
640	2794^m	648	2857^m	720	2906^m
728	2920^m	819	3376^ℓ	856	3459^m
928	3508^m	968	3522^m	1000	3557^m
1056	3571^m	1144	3606^m	1208	3641^m
1240	3655^m	1280	3669^m	1360	3683^m
1600	3704^m	1800	3880^ℓ	2560	3984^m
2880	4104^m	3200	4202^ℓ	3240	4280^ℓ
4608	4376^m	5120	4474^m	5184	4537^m



5696	4586^m	5760	4635^m	5824	4698^m	6464	5154^m	6552	5168^m	6848	5251^m
7128	5300^m	7424	5335^m	7616	5349^m	7744	5398^m	8000	5433^m	8256	5447^m
8448	5461^m	8712	5496^m	8896	5531^m	9152	5545^m	9504	5580^m	9664	5615^m
9920	5629^m	10000	5643^m								

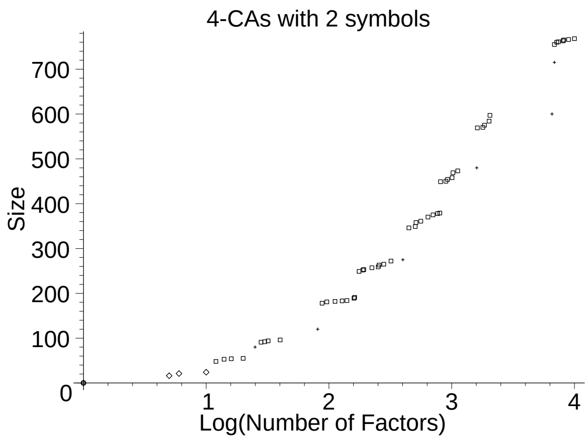
10	729^o	12	$1329^\downarrow$	20	1377^ℓ
41	1449^v	90	2025^m	113	2169^v
117	2865^m	225	2889^v	369	3321^m
410	3474^ℓ	810	3897^m	891	4041^m
900	4105^m	1017	4233^m	1130	4842^ℓ
1161	4953^m	1251	5017^m	1260	5065^m
1341	5081^m	1512	5145^m	1629	5209^m
1638	5257^m	1755	5273^m	1764	5321^m
2025	5337^m	2250	5562^ℓ	3321	5769^m
3690	5922^m	4059	6066^ℓ	4100	6147^ℓ
7290	6345^m	8019	6489^m	8100	6553^m
8829	6681^m	8910	6745^m	9000	6809^m



5.2 Tables for Strength Four

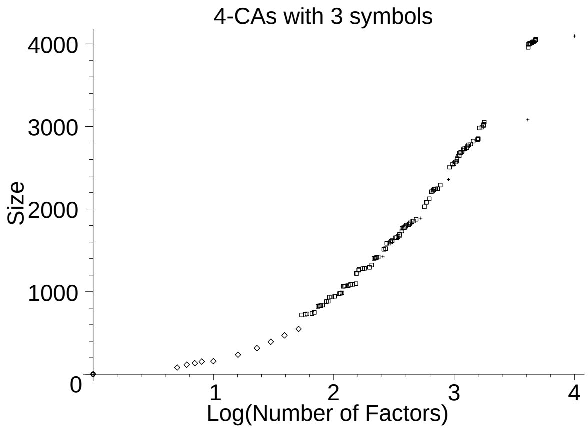
Here we report similar results for strength four; the only table of which we are aware appears in [18], and treats only $k \leq 10$.

5	16^o	6	21^u	10	24^f
12	48^r	14	53^r	16	54^r
20	55^r	25	80^q	28	91^r
30	92^r	32	94^r	40	96^r
81	120^q	88	178^r	96	181^r
112	182^r	128	183^r	140	184^r
160	189^r	162	191^r	176	249^r
189	252^r	192	253^r	224	257^r
252	259^r	256	263^r	280	265^r
320	272^r	400	275^q	448	346^r
504	349^r	512	358^r	560	361^r
640	370^r	704	375^r	768	378^r



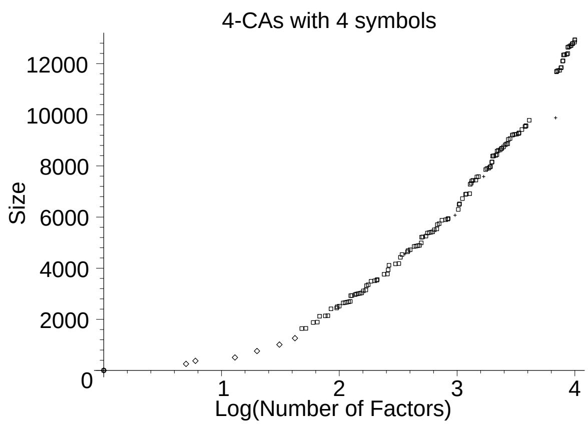
800	379^r	810	449^r	896	450^r	924	454^r	1008	458^r	1024	469^r
1120	473^r	1600	480^q	1620	569^r	1792	570^r	1848	575^r	2016	584^r
2048	597^r	6561	600^q	6859	715^h	6864	755^r	7168	760^r	7392	761^r
8064	763^r	8192	765^r	8960	766^r	10000	768^r				

5	81^o	6	115^s	7	133^s
8	153^s	10	159^v	16	237^v
23	315^v	30	393^v	39	471^v
51	549^v	54	718^r	58	726^r
60	730^r	66	735^r	69	749^r
74	822^r	76	828^r	78	832^r
81	837^r	87	881^r	90	885^r
92	934^r	96	936^r	102	944^r
111	975^r	114	981^r	117	985^r
120	1065^r	123	1067^r	126	1069^r
129	1071^r	132	1073^r	138	1087^r
144	1089^r	153	1097^r	154	1221^r



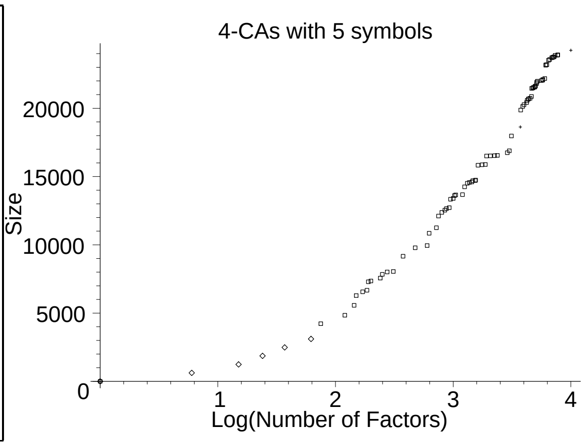
156	1222^r	161	1260^r	162	1268^r	174	1278^r	180	1282^r	198	1295^r
207	1323^r	216	1402^r	222	1406^r	225	1412^r	228	1416^r	234	1420^r
256	1422^q	261	1513^r	270	1521^r	276	1586^r	288	1588^r	297	1602^r
300	1610^r	306	1618^r	324	1651^r	333	1655^r	342	1667^r	351	1675^r
352	1693^r	368	1735^r	369	1769^r	378	1773^r	387	1777^r	396	1793^r
400	1805^r	420	1811^r	426	1821^r	432	1833^r	447	1847^r	459	1855^r
484	1877^r	529	1890^h	567	2028^r	588	2081^r	594	2083^r	621	2125^r
648	2210^r	666	2218^r	675	2232^r	684	2240^r	702	2244^r	729	2246^r
768	2290^r	900	2358^q	918	2508^r	972	2543^r	999	2551^r	1026	2569^r
1053	2581^r	1056	2601^r	1058	2619^r	1080	2643^r	1104	2645^r	1107	2679^r
1134	2685^r	1136	2687^r	1161	2691^r	1188	2713^r	1200	2729^r	1224	2731^r
1260	2739^r	1278	2743^r	1296	2763^r	1320	2777^r	1377	2787^r	1440	2823^r
1521	2826^q	1566	2842^r	1584	2844^r	1587	2852^r	1620	2982^r	1701	2992^r
1755	3013^r	1764	3025^r	1782	3051^r	4096	3081^h	4131	3959^r	4158	3995^r
4200	4003^r	4266	4005^r	4320	4009^r	4428	4016^r	4500	4024^r	4563	4026^r
4698	4042^r	4752	4046^r	4761	4054^r	10000	4095^h				

5	256^o	6	375^s	13	508^v
20	760^v	31	1012^v	42	1264^v
48	1639^r	52	1648^r	60	1878^r
65	1890^r	68	2119^r	76	2136^r
80	2142^r	85	2412^r	95	2444^r
96	2489^r	100	2514^r	108	2641^r
112	2656^r	116	2671^r	120	2686^r
124	2701^r	125	2925^r	128	2933^r
136	2968^r	140	2988^r	145	3003^r
150	3018^r	155	3033^r	160	3112^r
168	3148^r	170	3315^r	176	3351^r
186	3488^r	200	3507^r	208	3532^r



210	3555^r	240	3762^r	256	3774^r	260	3939^r	264	4113^r	300	4169^r
320	4181^r	330	4425^r	341	4535^r	361	4560^h	380	4643^r	384	4688^r
400	4722^r	432	4849^r	448	4864^r	464	4879^r	480	4894^r	496	4990^r
500	5214^r	512	5222^r	544	5257^r	560	5376^r	580	5391^r	600	5406^r
620	5421^r	640	5500^r	672	5536^r	680	5703^r	704	5739^r	744	5876^r
800	5895^r	832	5920^r	840	5943^r	961	6072^h	1024	6297^r	1040	6492^r
1050	6515^r	1110	6722^r	1180	6890^r	1200	6902^r	1280	6914^r	1292	7280^r
1320	7327^r	1332	7411^r	1364	7437^r	1444	7447^r	1472	7568^r	1520	7583^r
1681	7584^h	1748	7854^r	1792	7900^r	1856	7915^r	1900	7957^r	1920	7972^r
1968	8143^r	1984	8158^r	2000	8382^r	2036	8390^r	2048	8392^r	2128	8412^r
2176	8442^r	2185	8579^r	2240	8610^r	2320	8625^r	2375	8676^r	2400	8691^r
2480	8742^r	2560	8821^r	2624	8857^r	2688	8866^r	2720	9033^r	2816	9069^r
2944	9206^r	2976	9215^r	3072	9234^r	3200	9243^r	3328	9268^r	3360	9291^r
3552	9420^r	3776	9540^r	3840	9558^r	3844	9573^r	4096	9783^r	6859	9880^h
6984	11682^r	6992	11697^r	7168	11728^r	7424	11743^r	7600	11836^r	7680	11851^r
7872	12097^r	7936	12112^r	8000	12336^r	8140	12344^r	8192	12346^r	8512	12366^r
8704	12396^r	8736	12638^r	8740	12648^r	8960	12669^r	9216	12705^r	9280	12709^r
9480	12774^r	9600	12789^r	9920	12840^r	9988	12919^r	10000	12934^r		

6	625^o	15	1245^v	24	1865^v
37	2485^v	62	3105^v	75	4225^r
120	4845^r	144	5571^r	150	6287^r
170	6557^r	185	6675^r	190	7295^r
200	7357^r	240	7565^r	250	7837^r
275	8013^r	310	8045^r	375	9165^r
475	9785^r	600	9945^r	625	10851^r
720	11251^r	750	12107^r	800	12377^r
850	12537^r	875	12655^r	925	12719^r
950	13339^r	1000	13401^r	1025	13609^r
1050	13657^r	1200	13673^r	1250	14249^r
1320	14509^r	1375	14573^r	1440	14605^r



1470	14697^r	1540	14713^r	1550	14737^r	1625	15833^r	1760	15865^r	1875	15881^r
1920	16501^r	2070	16517^r	2250	16533^r	2375	16549^r	2880	16745^r	3000	16885^r
3125	17971^r	3721	18630^h	3750	19869^r	3900	20139^r	4000	20265^r	4200	20421^r
4250	20573^r	4350	20667^r	4375	20691^r	4500	20731^r	4625	20865^r	4650	21461^r
4750	21485^r	4800	21523^r	4920	21547^r	4950	21599^r	5000	21623^r	5100	21807^r
5125	21909^r	5200	21985^r	5610	22033^r	5760	22057^r	5780	22109^r	6000	22179^r
6120	23155^r	6125	23179^r	6250	23195^r	6460	23515^r	6600	23573^r	6875	23701^r
7020	23733^r	7080	23749^r	7200	23765^r	7350	23873^r	7700	23889^r	7750	23913^r
10000	24245^h										

6 Concluding Remarks

The recursive constructions for strength three developed here provide a useful complement to that in [10]. More importantly, the recursive constructions for strength four provide numerous powerful techniques for the construction of covering arrays. The existence tables demonstrate the utility of computational search for small arrays combined with flexible recursive constructions. The constructions using perfect hash families and Turán graphs provide some of the best bounds as the number of columns (factors) increases, but currently do not exhibit the generality of the Roux-type constructions developed here.

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