A recursive construction for simple t-designs using resolutions

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Abstract

This work presents a recursive construction for simple t-designs using resolutions of the ingredient designs. The result extends a construction of t-designs in our recent paper [39]. Essentially, the method in [39] describes the blocks of a constructed design as a collection of block unions from a number of appropriate pairs of disjoint ingredient designs. Now, if some pairs of these ingredient t-designs have both suitable s-resolutions, then we can define a distance mapping on their resolution classes. Using this mapping enables us to have more possibilities for forming blocks from those pairs. The method makes it possible for constructing many new simple t-designs. We give some application results of the new construction.

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1 Introduction

In a recent paper [39] we have presented a recursive method for constructing simple t-designs for arbitrary t. The method is of combinatorial nature since it requires finding solutions for the indices of the ingredient designs that satisfy a certain set of equalities. In essence, the core of the construction is that the blocks of a constructed design are built as a collection of block unions from a number of appropriate pairs of disjoint ingredient designs. In particular, when a pair of ingredient designs is used, we take as new blocks the unions of all the pairs of blocks in the two ingredient designs. For the sake of simplicity we refer to this construction method as the basic method or the basic construction.

In the present paper we describe an extension of the basic construction by assuming that a subset of pairs of ingredient designs have suitable resolutions. For those given pairs we may define a distance mapping on their resolution classes. By using this mapping we have more possibilities for forming blocks from those pairs other than taking the unions of all possible pairs of blocks in the ingredient designs. This construction actually extends the basic construction since many new simple t-designs can only be constructed with the new method. The crucial point of this extension is the use of s-resolutions for t-designs. The concept of s-resolutions may be viewed as a generalization of the notion of parallelisms, which may be termed as $(1,1)$-resolutions, i.e. the blocks of the t-design can be partitioned into classes of mutually disjoint blocks such that every point is in exactly one block of each class. To date very little is known about s-resolutions for t-designs when $s \geq 2$, except for the trivial t-designs. In this case, an s-resolution of the trivial t-design turns out to be a large set of s-designs. A great deal of results about large sets of s-designs have been achieved by many researchers, see the references below. We will describe our construction in terms of s-resolutions for t-designs in general. However we will restrict its applications just for the case where pairs of trivial designs are used and each has a suitable large set. Even with this limitation we find that the construction using resolutions still possesses its strength since it produces many simple t-designs.

It is worthwhile to emphasize that constructing simple t-designs for large t is a challenging problem in design theory. There are several major approaches to the problem. These include constructing t-designs from large sets of t-designs, for instance [1, 18, 13, 16, 19, 21, 23, 24, 25, 32, 33, 34, 41]; constructing t-designs by using prescribed automorphism groups, for example [2, 3, 6, 7, 8, 9, 10, 14, 20, 22, 26, 29]; or contructing t-designs via recursive construction methods, see for instance [15, 17, 27, 31, 30, 36, 37, 38, 39, 40].

2 Preliminaries

We recall some basic definitions. A t-design, denoted by t-(v,k,λ), is a pair $(X,\mathfrak{B})$, where X is a v-set of *points* and $\mathfrak{B}$ is a collection of k-subsets, called *blocks*, of X having the property that every t-set of X is a subset of exactly λ blocks in $\mathfrak{B}$. The parameter λ is called the *index* of the design. A t-design is called *simple* if no two blocks are identical i.e. no block of $\mathcal{B}$ is repeated; otherwise, it is called *non-simple* (i.e. $\mathfrak{B}$ is a multiset). It can be shown by simple counting that a t-(v,k,λ) design is an s-(v,k,λ_s) design for $0 \leq s \leq t$, where $\lambda_s = \lambda\binom{v-s}{t-s}/\binom{k-s}{t-s}$. Since λ_s is an integer, necessary conditions for the parameters of a $t-$design are $\binom{k-s}{t-s}|\lambda\binom{v-s}{t-s}$, for $0 \leq s \leq t$. For given t, v and k, we denote by $\lambda_{\min}(t,k,v)$, or $\lambda_{\min}$ for short, the smallest positive integer such that these conditions are satisfied for all $0 \leq s \leq t$. By complementing each block in X of a t-(v,k,λ) design, we obtain a t-$(v, v-k, \lambda^*)$ design with $\lambda^* = \lambda\binom{v-k}{t}/\binom{k}{t}$, hence we shall assume that $k \leq v/2$. The largest value for λ for which a simple t-(v,k,λ) design exists is denoted by $\lambda_{\max}$ and we have $\lambda_{\max} = \binom{v-t}{k-t}$. The simple t-$(v,k,\lambda_{\max})$ design is called the *complete* design or the *trivial* design. A t-$(v,k,1)$ design is called a *t-Steiner system.*

We refer the reader to [5, 12] for more information about designs.

Definition 2.1 *A t-(v,k,λ)-design $(X,\mathfrak{B})$ is said to be (s,τ)-resolvable with $0 < s < t$, if its block set $\mathfrak{B}$ can be partitioned into N classes $\mathfrak{A}_1,\ldots,\mathfrak{A}_N$ such that $(X,\mathfrak{A}_i)$ is* *an s-(v,k,τ) design for all $i=1,\dots,N$. Each $\mathfrak{A}_i$ is called a resolution class. We also say that a t-(v,k,λ)-design has an s-resolution, if it is (s,τ)-resolvable for a certain τ.*

It is worth noting that the concept of resolvability (i.e. $(1,1)$-resolvability) for BIBD introduced by Bose in 1942 [11] was generalized by Shrikhande and Raghavarao to τ-resolvability (i.e. $(1,\tau)$-resolvability) for BIBD in 1963 [28]. A definition of (s,λ)-resolvability for t-designs with $t\geq 3$ may be found in [4]. In that paper Baker shows that the Steiner quadruple system 3-$(4^m,4,1)$ constructed from an even dimensional affine space over the field of two elements has a $(2,1)$-resolution. Also, Teirlinck shows for example that there exists a 2-resolvable 3-$(2p^n+2,4,1)$ design with $p\in\{7,31,127\}$, for any positive integer n, [35].

To date, very little is known about s-resolutions of non-trivial t-(v,k,λ) designs for $t\geq 3$ and $s\geq 2$. Here are examples with $t=4$ and $s=3$. In [2], Alltop has shown that there exists a simple 4-$(q+1,5,5)$ design for every $q=2^n$, $n\geq 5$, n odd. This 4-design $(X,\mathfrak{B})$ is constructed by using the group PGL$(2,q)$, which acts sharply 3-transitively on the projective line $X=\mathrm{GF}(q)\cup\{\infty\}$. The block set $\mathfrak{B}$ is a disjoint union of $(q-2)/6$ orbits of 5-sets of X under PGL$(2,q)$. Each orbit forms a 3-$(q+1,5,15)$ design. Hence each 4-$(q+1,5,5)$ design in the Alltop's family has a $(3,15)$-resolution.

When $(X,\mathfrak{B})$ is the trivial t-$(v,k,\binom{v-t}{k-t})$ design, then an (s,τ)-resolution of $(X,\mathfrak{B})$ is called a *large set*. Thus, a large set is a partition of the complete t-$(v,k,\binom{v-t}{k-t})$ design into s-(v,k,τ) designs, and is denoted by LS$[N](s,k,v)$, where $N=\binom{v-s}{k-s}/\tau$ is the number of resolution classes in the partition.

We define a distance on the resolution classes of a t-design as follows.

Definition 2.2 *Let D be a t-(v,k,λ) design admitting an (s,τ)-resolution with $\mathfrak{A}_1,\dots,\mathfrak{A}_N$ as resolution classes. Define a distance between any two classes $\mathfrak{A}_i$ and $\mathfrak{A}_j$ by $d(\mathfrak{A}_i,\mathfrak{A}_j)=$* $\min\{|i-j|,N-|i-j|\}$.

2.1 The basic construction

In this section, we summarize the basic construction as described in [39]. This preparation is necessary for the description of the construction using resolution in the next section.

We first give notation and definitions. Let t, v, k be non-negative integers such that $v\geq k\geq t\geq 0$. Let X be a v-set and let $X=X_1\cup X_2$ be a partition of X (i.e $X_1\cap X_2=\emptyset$) with $|X_1|=v_1$ and $|X_2|=v_2$.

The parameter set t-$(v_2,j,\bar{\lambda}_t^{(j)})$ for a design indicates that the point set of the design is X_2. Also, a design defined on the point set X_2 is denoted by $\bar{D}=(X_2,\bar{\mathfrak{B}})$.

(i) For $i=0,\dots,t$, let $D_i=(X_1,\mathfrak{B}^{(i)})$ be the complete i-$(v_1,i,1)$ design. For $i=t+1,\dots,k$, let $D_i=(X_1,\mathfrak{B}^{(i)})$ be a simple t-$(v_1,i,\lambda_t^{(i)})$ design.

(ii) Similarly, for $i = 0, \dots, t$, let $\bar{D}_i = (X_2, \bar{\mathfrak{B}}^{(i)})$ be the complete i-$(v_2, i, 1)$ design. And for $i = t+1, \dots, k$, let $\bar{D}_i = (X_2, \bar{\mathfrak{B}}^{(i)})$ be a simple t-$(v_2, i, \bar{\lambda}_t^{(i)})$ design.

(iii) Two degenerate cases for designs occur when either $k = t = 0$ or $v = k$. The first case $k = t = 0$ gives an "empty" design, denoted by $\emptyset$, however we use the convention that the number of blocks of the empty design is 1 (i.e. the unique block is the empty block). The second case $v = k$ gives a degenerate k-design having just 1 block consisting of all v points. Thus, in these two extreme cases the number of blocks of the designs is always 1.

(iv) We denote by $T_{(r,t-r)}$ a t-subset T of X with $|T \cap X_1| = r$ and hence $|T \cap X_2| = t - r$, for $r = 0, \dots, t$. It is clear that any t-subset of X is a $T_{(r,t-r)}$ set for some $r \in \{0, \dots, t\}$.

(v) Let X be a finite set and let $u \in \{0, 1\}$. The notation $X \times [u]$ has the following meaning. $X \times [0]$ is the empty set $\emptyset$, and $X \times [1] = X$.

The basic construction in [39] is as follows.

Consider $(k+1)$ pairs of simple designs $(D_i, \bar{D}_{k-i})$ for $i = 0, \dots, k$, where $D_i = (X_1, \mathfrak{B}^{(i)})$ is a simple t-$(v_1, i, \lambda_t^{(i)})$ design and $\bar{D}_{k-i} = (X_2, \bar{\mathfrak{B}}^{(k-i)})$ a simple t-$(v_2, k - i, \bar{\lambda}_t^{(k-i)})$ design, as defined above. For each pair $(D_i, \bar{D}_{k-i})$ define

$$\mathfrak{B}_{(i,k-i)} := \{B = B_i \cup \bar{B}_{k-i} \ \mid \ B_i \in \mathfrak{B}^{(i)}, \bar{B}_{k-i} \in \bar{\mathfrak{B}}^{(k-i)}\}.$$

Define

$$\mathfrak{B} := \mathfrak{B}_{(0,k)} \times [u_0] \cup \mathfrak{B}_{(1,k-1)} \times [u_1] \cup \cdots \cup \mathfrak{B}_{(k-1,1)} \times [u_{k-1}] \cup \mathfrak{B}_{(k,0)} \times [u_k],$$

where $u_i \in \{0, 1\}$, for $i = 0, \dots, k$.

It should be remarked that the notation $\mathfrak{B}_{(i,k-i)} \times [u_i]$, as defined in (v) above, indicates that either we have an empty set $\emptyset$ (when $u_i = 0$) or the set $\mathfrak{B}_{(i,k-i)}$ itself (when $u_i = 1$). The empty set case means that the pair $(D_i, \bar{D}_{k-i})$ is not used and the other case means the pair $(D_i, \bar{D}_{k-i})$ is used.

It can be shown that for a given t-set $T_{(r,t-r)}$ of X the number of blocks in $\mathfrak{B}$ containing $T_{(r,t-r)}$ is equal to

$$L_{r,t-r} \quad := \quad \sum_{i=0}^{k} u_i . \lambda_r^{(i)} . \bar{\lambda}_{t-r}^{(k-i)}.$$

Therefore, if

$$L_{0,t} = L_{1,t} = L_{2,t-2} = \cdots = L_{t,0} := \Lambda,$$

where Λ is a positive integer, then $(X, \mathfrak{B})$ forms a simple t-design with parameters t-(v, k, Λ).

We record the basic construction in the following theorem.

Theorem 2.1 (Basic construction) *Let v, k, t be integers with $v > k > t \geq 2$. Let X be a v-set and let $X = X_1 \cup X_2$ be a partition of X with $|X_1| = v_1$ and $|X_2| = v_2$. Let $D_i = (X_1, \mathfrak{B}^{(i)})$ be the complete i-$(v_1, i, 1)$ design for $i = 0, \ldots, t$ and let $D_i = (X_1, \mathfrak{B}^{(i)})$ be a simple t-$(v_1, i, \lambda_t^{(i)})$ design for $i = t+1, \ldots, k$. Similarly, let $\bar{D}_i = (X_2, \bar{\mathfrak{B}}^{(i)})$ be the complete i-$(v_2, i, 1)$ design for $i = 0, \ldots, t$, and let $\bar{D}_i = (X_2, \bar{\mathfrak{B}}^{(i)})$ be a simple t-$(v_2, i, \bar{\lambda}_t^{(i)})$ design for $i = t+1, \ldots, k$. Define*

$$\mathfrak{B} = \mathfrak{B}_{(0,k)} \times [u_0] \cup \mathfrak{B}_{(1,k-1)} \times [u_1] \cup \cdots \cup \mathfrak{B}_{(k-1,1)} \times [u_{k-1}] \cup \mathfrak{B}_{(k,0)} \times [u_k],$$

where

$$\mathfrak{B}_{(i,k-i)} = \{B = B_i \cup \bar{B}_{k-i} \mid B_i \in \mathfrak{B}^{(i)}, \bar{B}_{k-i} \in \bar{\mathfrak{B}}^{(k-i)}\}.$$

Assume that

$$L_{0,t} = L_{1,t-1} = L_{2,t-2} = \cdots = L_{t,0} := \Lambda, \tag{1}$$

for a positive integer Λ, where

$$L_{r,t-r} = \sum_{i=0}^{k} u_i . \lambda_r^{(i)} . \bar{\lambda}_{t-r}^{(k-i)}, \tag{2}$$

$r = 0, \ldots, t$, and $u_i \in \{0, 1\}$, for $i = 0, \ldots, k$. Then $(X, \mathfrak{B})$ is a simple t-(v, k, Λ) design.

3 The construction using resolutions

In this section we describe a recursive construction of simple t-designs using resolutions. Note that in the basic construction, if a pair $(D_i, \bar{D}_{k-i})$ is used in the construction (i.e. $u_i = 1$), then the new blocks formed by this pair consist of taking the union of each block of D_i with each block of $\bar{D}_{k-i}$. The crucial idea of the construction using resolutions is that if D_i and $\bar{D}_{k-i}$ have appropriate s_1- and s_2-resolutions with the same number of resolution classes, then the new blocks are formed according to the distance mapping defined on the resolution classes of D_i and $\bar{D}_{k-i}$ rather than taking the unions of each block of D_i with each block of $\bar{D}_{k-i}$.

In the following we go into detail of the construction. We make use of the notation and definitions for the basic construction in the previous section. When for a certain $i \in \{0, \ldots, k\}$ the t-$(v_1, i, \lambda_t^{(i)})$ design $D_i = (X_1, \mathfrak{B}^{(i)})$ has an s_i-resolution, i.e. D_i can be partitioned into N_i disjoint $(X_1, \mathfrak{A}_h^{(i)})$ designs with parameters s_i-$(v_1, i, \lambda_{s_i}^{*(i)})$, $s_i < t$, then we write

$$\mathfrak{B}^{(i)} = \bigcup_{h=1}^{N_i} \mathfrak{A}_h^{(i)},$$

where

$$N_i = \lambda_t^{(i)} \binom{v_1 - s_i}{t - s_i} / \lambda_{s_i}^{*(i)} \binom{i - s_i}{t - s_i}.$$

Similarly, we write

$$\bar{\mathfrak{B}}^{(k-i)} = \bigcup_{h=1}^{\bar{N}_{k-i}} \bar{\mathfrak{A}}_h^{(i)},$$

when the blocks of a t-$(v_2, k-i, \bar{\lambda}_t^{(k-i)})$ design $\bar{D}_{k-i} = (X_2, \bar{\mathfrak{B}}^{(k-i)})$ can be partitioned into $\bar{N}_{k-i}$ disjoint $(X_2, \bar{\mathfrak{A}}_h^{(k-i)})$ designs with parameters s_{k-i}-$(v_2, k-i, \bar{\lambda}^{*(k-i)}_{s_{k-i}})$, where

$$\bar{N}_{k-i} = \bar{\lambda}_t^{(k-i)} \binom{v_2 - s_{k-i}}{t - s_{k-i}} / \bar{\lambda}^{*(k-i)}_{s_{k-i}} \binom{k-i-s_{k-i}}{t-s_{k-i}}$$

is the number of s_{k-i}-resolution classes.

Let $K = \{(0,k), (1,k-1), \ldots, (k-1,1), (k,0)\}$. Assume there exists a subset $R \subseteq K$ such that if $(i, k-i) \in R$, then D_i and $\bar{D}_{k-i}$ have an s_i-resolution of size N_i and an s_{k-i}-resolution of size $\bar{N}_{k-i}$, respectively, satisfying the following conditions.

(i) $N_i = \bar{N}_{k-i}$,

(ii) $s_i + s_{k-i} \geq 2\lfloor \frac{t}{2} \rfloor$.

The construction consists of building two types of blocks.

(1) For each pair $(i, k-i) \in K \setminus R$ form a subset of new blocks $\mathfrak{B}_{(i,k-i)}$ from the pair $(D_i, \bar{D}_{k-i})$ as

$$\mathfrak{B}_{(i,k-i)} := \{B = B_i \cup \bar{B}_{k-i} \ \mid \ B_i \in \mathfrak{B}^{(i)}, \bar{B}_{k-i} \in \bar{\mathfrak{B}}^{(k-i)}\}.$$

(2) For each pair $(i, k-i) \in R$ form a subset of new blocks $\mathfrak{B}^*_{(i,k-i)}$ from $(D_i, \bar{D}_{k-i})$ by using an s_i-resolution of D_i and an s_{k-i}-resolution of $\bar{D}_{k-i}$ as follows.

$$\mathfrak{B}^*_{(i,k-i)} := \{B_i \cup \bar{B}_{k-i} \ \mid \ B_i \in \mathfrak{A}_h^{(i)}, \bar{B}_{k-i} \in \bar{\mathfrak{A}}_j^{(k-i)}, \ \varepsilon_i \leq d(\mathfrak{A}_h^{(i)}, \mathfrak{A}_j^{(i)}) \leq w_i, \varepsilon_i = 0, 1; \ w_i \leq \lfloor \frac{N_i}{2} \rfloor\}.$$

Further, define

$$z_i := (2w_i + 1 - \varepsilon_i), \text{ if } \ w_i < \frac{N_i}{2}, \text{ and } z_i := (2w_i - \varepsilon_i), \text{ if } \ w_i = \frac{N_i}{2}.$$

Note that w_i and z_i are considered as variables.

Now, let $T_{(r,t-r)}$ be a t-set of X for $r = 0, \ldots, t$. According to the property of s_i and s_{k-i} one of the following cases has to occur.

(a) $r \leq s_i$ and $t - r \leq s_{k-i}$. Then $T_{(r,t-r)}$ is contained in

$$\Lambda^{*(i,k-i)}_{r,t-r} = \lambda_r^{*(i)}.\bar{\lambda}^{*(k-i)}_{t-r}.N_i.z_i$$

blocks of $\mathfrak{B}^*_{(i,k-i)}$.

(b) $r \leq s_i$ and $t - r > s_{k-i}$. Then $T_{(r,t-r)}$ is contained in

$$\Lambda^{*(i,k-i)}_{r,t-r} = \lambda^{*(i)}_r.\bar{\lambda}^{(k-i)}_{t-r}.z_i$$

blocks of $\mathfrak{B}^*_{(i,k-i)}$.

(c) $r > s_i$ and $t - r \leq s_{k-i}$. Then $T_{(r,t-r)}$ is contained in

$$\Lambda^{*(i,k-i)}_{r,t-r} = \lambda^{(i)}_r.\bar{\lambda}^{*(k-i)}_{t-r}.z_i$$

blocks of $\mathfrak{B}^*_{(i,k-i)}$.

It is straightforward to verify the values of $\Lambda^{*(i,k-i)}_{r,t-r}$ for the cases (a), (b) and (c) above. In case (a) each r-subset of X_1 is contained in $\lambda^{*(i)}_r$ blocks of $\mathfrak{A}^{(i)}_h$ and each $(t-r)$-subset of X_2 in $\bar{\lambda}^{*(k-i)}_{t-r}$ blocks of $\bar{\mathfrak{A}}^{(k-i)}_j$. Thus each pair $(\mathfrak{A}^{(i)}_h, \bar{\mathfrak{A}}^{(k-i)}_j)$ contributes $\lambda^{*(i)}_r.\bar{\lambda}^{*(k-i)}_{t-r}$ blocks to $\mathfrak{B}^*_{(i,k-i)}$. Now each of the N_i resolution classes $\mathfrak{A}^{(i)}_1, \ldots, \mathfrak{A}^{(i)}_{N_i}$ is combined with z_i resolution classes of $\bar{\mathfrak{A}}^{(k-i)}_1, \ldots, \bar{\mathfrak{A}}^{(k-i)}_{N_i}$, therefore we have $\Lambda^{*(i,k-i)}_{r,t-r} = \lambda^{*(i)}_r.\bar{\lambda}^{*(k-i)}_{t-r}.N_i.z_i$

In case (b) each r-subset of X_1 is contained in $\lambda^{*(i)}_r$ blocks of $\mathfrak{A}^{(i)}_h$ and each $(t-r)$-subset of X_2 in $\bar{\lambda}^{(k-i)}_{t-r}$ blocks of $\bar{\mathfrak{B}}^{(k-i)}$. These blocks are distributed in the N_i resolution classes $\bar{\mathfrak{A}}^{(k-i)}_1, \ldots, \bar{\mathfrak{A}}^{(k-i)}_{N_i}$. Each class $\bar{\mathfrak{A}}^{(k-i)}_j$ is combined z_i times with $\mathfrak{A}^{(i)}_h$. Hence, in this case, the contribution of the blocks to $\mathfrak{B}^*_{(i,k-i)}$ is $\Lambda^{*(i,k-i)}_{r,t-r} = \lambda^{*(i)}_r.\bar{\lambda}^{(k-i)}_{t-r}.z_i$.

The case (c) is similar to case (b).

Define

$$\mathfrak{B} := \bigcup_{(i,k-i)\in R} \mathfrak{B}^*_{(i,k-i)} \times [u_i] \ \cup \bigcup_{(i,k-i)\in K\setminus R} \mathfrak{B}_{(i,k-i)} \times [u_i],$$

with $u_i \in \{0, 1\}$, $i = 0, \ldots, k$.

The above presentation can be summarized as follows. Let $T_{(r,t-r)}$ be a t-subset of X for $r = 0, \ldots, t$. The number of blocks in $\mathfrak{B}_{(i,k-i)}$ containing $T_{(r,t-r)}$, for all $(i, k-i) \in K \setminus R$, is then

$$\sum_{(i,k-i)\in K\setminus R} u_i.\Lambda^{(i,k-i)}_{(r,t-r)} = \sum_{(i,k-i)\in K\setminus R} u_i.\lambda^{(i)}_r.\bar{\lambda}^{(k-i)}_{t-r}.$$

The number of blocks in $\mathfrak{B}^*_{(i,k-i)}$ containing $T_{(r,t-r)}$, for all $(i, k-i) \in R$, is then

$$\sum_{(i,k-i)\in R} u_i.\Lambda^{*(i,k-i)}_{(r,t-r)},$$

where

$$\Lambda^{*(i,k-i)}_{(r,t-r)} = \begin{cases} \lambda^{*(i)}_r.\bar{\lambda}^{*(k-i)}_{t-r}.N_i.z_i & \text{if } r \leq s_i,\ t-r \leq s_{k-i}, \\ \lambda^{*(i)}_r.\bar{\lambda}^{(k-i)}_{t-r}.z_i & \text{if } r \leq s_i,\ t-r > s_{k-i}, \\ \lambda^{(i)}_r.\bar{\lambda}^{*(k-i)}_{t-r}.z_i & \text{if } r > s_i,\ t-r \leq s_{k-i}. \end{cases}$$

It follows that the number of blocks in $\mathfrak{B}$ containing $T_{r,t-r}$ is equal to

$$L_{r,t-r} := \sum_{(i,k-i)\in R} u_i.\Lambda^{*(i,k-i)}_{(r,t-r)} + \sum_{(i,k-i)\in K\setminus R} u_i.\Lambda^{(i,k-i)}_{(r,t-r)}.$$

Since any t-subset of X is of form $T_{r,t-r}$ for some $r \in \{0,\ldots,t\}$, we see that if

$$L_{0,t} = L_{1,t-1} = \cdots = L_{t,0} := \Lambda$$

for a positive integer Λ, then $(X,\mathfrak{B})$ forms a simple t-design with parameters t-(v,k,Λ).

We record the construction above in the following theorem.

Theorem 3.1 *Let v, k, t be integers with $v > k > t \geq 2$. Let X be a v-set and let $X = X_1 \cup X_2$ be a partition of X with $|X_1| = v_1$ and $|X_2| = v_2$. Let $D_i = (X_1, \mathfrak{B}^{(i)})$ be the complete i-$(v_1,i,1)$ design for $i = 0,\ldots,t$ and let $D_i = (X_1,\mathfrak{B}^{(i)})$ be a simple t-$(v_1,i,\lambda_t^{(i)})$ design for $i = t+1,\ldots,k$. Similarly, let $\bar{D}_i = (X_2,\bar{\mathfrak{B}}^{(i)})$ be the complete i-$(v_2,i,1)$ design for $i = 0,\ldots,t$, and let $\bar{D}_i = (X_2,\bar{\mathfrak{B}}^{(i)})$ be a simple t-$(v_2,i,\bar{\lambda}_t^{(i)})$ design for $i = t+1,\ldots,k$. Let $K = \{(0,k),(1,k-1),\ldots,(k-1,1),(k,0)\}$. Suppose there exists a subset $R \subseteq K$ such that for each $(i,k-i) \in R$, the designs D_i and $\bar{D}_{k-i}$ have an s_i-resolution with N_i classes and an s_{k-i}-resolution with $\bar{N}_{k-i}$ classes, respectively, satisfying the following conditions.*

(i) $N_i = \bar{N}_{k-i}$,

(ii) $s_i + s_{k-i} \geq 2\lfloor \frac{t}{2} \rfloor$.

Define

$$\mathfrak{B} = \bigcup_{(i,k-i)\in R} \mathfrak{B}^*_{(i,k-i)} \times [u_i] \ \cup \bigcup_{(i,k-i)\in K\setminus R} \mathfrak{B}_{(i,k-i)} \times [u_i],$$

for $u_i \in \{0,1\}$, $i = 0,\ldots,k$,

$$\mathfrak{B}^*_{(i,k-i)} := \{B_i \cup \bar{B}_{k-i} \ | \ B_i \in \mathfrak{A}^{(i)}_h, \bar{B}_{k-i} \in \bar{\mathfrak{A}}^{(k-i)}_j, \ \varepsilon_i \leq d(\mathfrak{A}^{(i)}_h, \mathfrak{A}^{(i)}_j) \leq w_i, \varepsilon_i = 0,1; \ w_i \leq \lfloor \frac{N_i}{2} \rfloor\},$$

with w_i as variable, where $\mathfrak{A}^{(i)}_1,\ldots,\mathfrak{A}^{(i)}_{N_i}$ are s_i-resolution classes of D_i, with $(X_1,\mathfrak{A}^{(i)}_h)$ as an s_i-$(v_1,i,\lambda^{*(i)}_{s_i})$ design; and $\bar{\mathfrak{A}}^{(k-i)}_1,\ldots,\bar{\mathfrak{A}}^{(k-i)}_{N_i}$ are s_{k-i}-resolution classes of $\bar{D}_{k-i}$, with $(X_2,\bar{\mathfrak{A}}^{(k-i)}_h)$ as an s_{k-i}-$(v_2,k-i,\bar{\lambda}^{*(k-i)}_{s_{k-i}})$ design; and

$$\mathfrak{B}_{(i,k-i)} := \{B = B_i \cup \bar{B}_{k-i} \ | \ B_i \in \mathfrak{B}^{(i)}, \bar{B}_{k-i} \in \bar{\mathfrak{B}}^{(k-i)}\}.$$

Define

$$L_{r,t-r} := \sum_{(i,k-i)\in R} u_i.\Lambda^{*(i,k-i)}_{(r,t-r)} + \sum_{(i,k-i)\in K\setminus R} u_i.\Lambda^{(i,k-i)}_{(r,t-r)},$$

for $r = 0,\ldots,t$, where

$$\Lambda^{*(i,k-i)}_{(r,t-r)} = \begin{cases} \lambda^{*(i)}_r.\bar{\lambda}^{*(k-i)}_{t-r}.N_i.z_i & \text{if } r \leq s_i, \ t-r \leq s_{k-i}, \\ \lambda^{*(i)}_r.\bar{\lambda}^{(k-i)}_{t-r}.z_i & \text{if } r \leq s_i, \ t-r > s_{k-i}, \\ \lambda^{(i)}_r.\bar{\lambda}^{*(k-i)}_{t-r}.z_i & \text{if } r > s_i, \ t-r \leq s_{k-i}. \end{cases}$$

with $z_i = (2w_i + 1 - \varepsilon_i)$, *if* $w_i < \frac{N_i}{2}$, *and* $z_i = (2w_i - \varepsilon_i)$, *if* $w_i = \frac{N_i}{2}$; *and*

$$\Lambda^{(i,k-i)}_{(r,t-r)} = \lambda^{(i)}_r . \bar{\lambda}^{(k-i)}_{t-r}.$$

Assume that

$$L_{0,t} = L_{1,t-1} = \cdots = L_{t,0} := \Lambda \qquad (3)$$

for a positive integer Λ*, then* $(X, \mathfrak{B})$ *is a simple* t*-*(v, k, Λ) *design.*

Remarks 3.1

1. In the basic construction the set $\mathfrak{B}_{(i,k-i)}$ of the new blocks is uniquely determined as the unions of all the pairs of blocks in D_i and $\bar{D}_{k-i}$. Whereas in the construction using resolutions in Theorem 3.1 the set $\mathfrak{B}^*_{(i,k-i)}$ is no longer unique. Its size varies according to the variable z_i.
2. Theorem 3.1 does not restrict to constructing simple t-designs. Obviously, if any of the ingredient designs is non-simple, then the construction will yield non-simple designs.

4 Applications

In this section we illustrate the construction in Theorem 3.1 through a number of examples which show the strength of the method.

In the following we will employ the notation from Chapter II.4 : t-Designs with $t \geq 3$ of the Handbook of Combinatorial Designs [12]. The parameter set t-(v, k, λ) of a design will be written as t-$(v, k, m\lambda_{\min})$. Since the supplement of a simple t-(v, k, λ) design is a t-$(v, k, \lambda_{\max} - \lambda)$ design, we usually consider simple t-(v, k, λ) designs with $\lambda \leq \lambda_{\max}/2$. Thus, the upper limit of m of a constructed design will be $\mathrm{LIM} = \lfloor \lambda_{\max}/(2\lambda_{\min}) \rfloor$. But, it should be remarked that, when an ingredient design with index λ is used, then λ can take on all possible values, i.e. $\lambda_{\min} \leq \lambda \leq \lambda_{\max}$.

4.1 Simple 5-$(38, k, \Lambda)$ designs with $k = 8, 9, 10$

We apply the construction in Theorem 3.1 to the cases $t = 5$, $v_1 = v_2 = 19$ and $k = 8, 9, 10$.

4.1.1 Simple 5-$(38, 8, \Lambda)$ designs

Here we show a detailed example to illustrate the construction.

Let $X = X_1 \cup X_2$ be a partition of the point set X with $|X| = 38$ into two subsets X_1 and X_2 with $|X_1| = |X_2| = 19$. For $i = 0, 1, 2, 3, 4, 5$ let $D_i = (X_1, \mathcal{B}^{(i)})$ be the complete i-$(19, i, \lambda^{(i)}_i) := i$-$(19, i, 1)$ design. For $i = 6, 7, 8$ let $D_i = (X_1, \mathcal{B}^{(i)})$ be a simple 5-$(19, i, \lambda^{(i)}_5)$ design. These designs have the following parameters.

- 5-$(19, 6, \lambda^{(6)}_5) =$ 5-$(19, 6, m2)$, $m = 1, 2, \ldots, 7$.

- $5\text{-}(19,7,\lambda_5^{(7)}) = 5\text{-}(19,7,m7)$, $m = 1,2,\dots,13$
- $5\text{-}(19,8,\lambda_5^{(8)}) = 5\text{-}(19,8,m28)$, $m = 1,2,\dots,13$

Correspondingly, let $\bar{D}_i = (X_2, \bar{\mathcal{B}}^{(i)})$ be simple designs defined on X_2. Here $K = \{(0,8),(1,7),(2,6),(3,5),(4,4),(5,3),(6,2),(7,1),(8,0)\}$.

It is known that the complete designs D_i and $\bar{D}_i$ for $i = 3,4,5$ have each a 2-resolution with the number of resolution classes $N_i = 17$, i.e. the large sets LS[17](2, i, 19), see for instance Chapter II.4 [12]. We choose

$$R = \{(3,5),(4,4),(5,3)\}.$$

Thus we have

- $\mathfrak{B}^{(3)} = \bigcup_{j=1}^{17} \mathfrak{A}_j^{(3)}$, where $(X_1, \mathfrak{A}_j^{(3)})$ is a $2\text{-}(19,3,\lambda_2^{*(3)}) = 2\text{-}(19,3,1)$ design, and $\lambda_3^{(3)} = 1,\ \lambda_2^{*(3)} = 1,\ \lambda_1^{*(3)} = 9,\ \lambda_0^{*(3)} = 57$;
- $\mathfrak{B}^{(4)} = \bigcup_{j=1}^{17} \mathfrak{A}_j^{(4)}$, where $(X_1, \mathfrak{A}_j^{(4)})$ is a $2\text{-}(19,4,\lambda_2^{*(4)}) = 2\text{-}(19,4,8)$ design and $\lambda_4^{(4)} = 1,\ \lambda_3^{(4)} = 16,\ \lambda_2^{*(4)} = 8,\ \lambda_1^{*(4)} = 48,\ \lambda_0^{*(4)} = 228$;
- $\mathfrak{B}^{(5)} = \bigcup_{j=1}^{17} \mathfrak{A}_j^{(5)}$, where $(X_1, \mathfrak{A}_j^{(i)})$ is a $2\text{-}(19,5,\lambda_2^{*(4)}) = 2\text{-}(19,5,40)$ design and $\lambda_5^{(5)} = 1,\ \lambda_4^{(5)} = 15,\ \lambda_3^{(5)} = 120,\ \lambda_2^{*(5)} = 40,\ \lambda_1^{*(5)} = 180,\ \lambda_0^{*(5)} = 684$;

Similarly, the complete designs $\bar{D}_i$ have the same 2-resolutions as D_i, each having $\bar{N}_i = 17$ resolution classes, for $i = 3,4,5$. Thus $\bar{\mathfrak{B}}^{(i)} = \bigcup_{j=1}^{17} \bar{\mathfrak{A}}_j^{(i)}$, and each $(X_2, \bar{\mathfrak{A}}_j^{(i)})$ is a $2\text{-}(19, i, \bar{\lambda}_2^{*(i)})$ design with $\bar{\lambda}_2^{*(i)} = \lambda_2^{*(i)}$.

We compute

$$L_{r,5-r} \;=\; \sum_{(i,8-i)\in R} u_i.\Lambda^{*(i,8-i)}_{(r,5-r)} + \sum_{(i,8-i)\in K\setminus R} u_i.\Lambda^{(i,8-i)}_{(r,5-r)},$$

for $r = 0,\dots,5$, and $u_i = 0,1$. If $(i,8-i) \in K \setminus R$, then

$$\Lambda^{(i,8-i)}_{(r,5-r)} = \lambda_r^{(i)}.\bar{\lambda}_{5-r}^{(8-i)}.$$

If $(i,8-i) \in R$, then the values of $\Lambda^{*(i,8-i)}_{(r,5-r)}$ are computed by using the formula

$$\Lambda^{*(i,k-i)}_{(r,t-r)} = \begin{cases} \lambda_r^{*(i)}.\bar{\lambda}_{t-r}^{*(k-i)}.N_i.z_i & \text{if } r \le s_i,\ t \le s_{k-i}, \\ \lambda_r^{*(i)}.\bar{\lambda}_{t-r}^{(k-i)}.z_i & \text{if } r \le s_i,\ t > s_{k-i}, \\ \lambda_r^{(i)}.\bar{\lambda}_{t-r}^{*(k-i)}.z_i & \text{if } r > s_i,\ t \le s_{k-i}. \end{cases}$$

Here we have

$$\begin{aligned} &\Lambda_{0,5}^{*(3,5)} = \lambda_0^{*(3)}.\bar{\lambda}_5^{(5)}.z_3 = 57z_3, \qquad \Lambda_{1,4}^{*(3,5)} = \lambda_1^{*(3)}.\bar{\lambda}_4^{(5)}.z_3 = 9 \times 15z_3, \\ &\Lambda_{2,3}^{*(3,5)} = \lambda_2^{*(3)}.\bar{\lambda}_3^{(5)}.z_3 = 120z_3, \qquad \Lambda_{3,2}^{*(3,5)} = \lambda_3^{(3)}.\bar{\lambda}_2^{*(5)}.z_3 = 40z_3, \\ &\Lambda_{4,1}^{*(3,5)} = \Lambda_{5,0}^{*(3,5)} = 0. \end{aligned}$$

$$\Lambda^{*(5,3)}_{0,5} = \Lambda^{*(5,3)}_{1,4} = 0, \qquad \Lambda^{*(5,3)}_{2,3} = \lambda^{*(5)}_{2}.\bar{\lambda}^{(3)}_{3}.z_5 = 40z_5,$$
$$\Lambda^{*(5,3)}_{3,2} = \lambda^{(5)}_{3}.\bar{\lambda}^{*(3)}_{2}.z_5 = 120z_5, \quad \Lambda^{*(5,3)}_{4,1} = \lambda^{(5)}_{4}.\bar{\lambda}^{*(3)}_{1}.z_5 = 15 \times 9z_5,$$
$$\Lambda^{*(5,3)}_{5,0} = \lambda^{(5)}_{5}.\bar{\lambda}^{*(3)}_{0}.z_5 = 57z_5.$$

$$\Lambda^{*(4,4)}_{0,5} = \Lambda^{*(4,4)}_{5,0} = 0, \qquad \Lambda^{*(4,4)}_{1,4} = \lambda^{*(4)}_{1}.\bar{\lambda}^{(4)}_{4}.z_4 = 48z_4,$$
$$\Lambda^{*(4,4)}_{2,3} = \lambda^{*(4)}_{2}.\bar{\lambda}^{(4)}_{3}.z_4 = 8 \times 16z_4, \quad \Lambda^{*(4,4)}_{3,2} = \lambda^{(4)}_{3}.\bar{\lambda}^{*(4)}_{2}.z_4 = 16 \times 8z_4,$$
$$\Lambda^{*(4,4)}_{4,1} = \lambda^{(4)}_{4}.\bar{\lambda}^{*(4)}_{1}.z_4 = 48z_4.$$

It follows that

$$\begin{aligned}
L_{0,5} &= u_0\bar{\lambda}^{(8)}_5 + u_1 19\bar{\lambda}^{(7)}_5 + u_2 171\bar{\lambda}^{(6)}_5 + u_3 57z_3,\\
L_{1,4} &= u_1 5\bar{\lambda}^{(7)}_5 + u_2 9 \times 15\bar{\lambda}^{(6)}_5 + u_3 9 \times 15z_3 + u_4 48z4,\\
L_{2,3} &= u_2 40\bar{\lambda}^{(6)}_5 + u_3 120z_3 + u_4 8 \times 16z_4 + u_5 40z5,\\
L_{3,2} &= u_6 40\lambda^{(6)}_5 + u_5 120z_5 + u_4 16 \times 8z4 + u_3 40z_3,\\
L_{4,1} &= u_7 5\lambda^{(7)}_5 + u_6 15 \times 9\lambda^{(6)}_5 + u_5 15 \times 9z_5 + u_4 48z4,\\
L_{5,0} &= u_8\lambda^{(8)}_5 + u_7 19\lambda^{(7)}_5 + u_6 171\lambda^{(6)}_5 + u_5 57z_5.
\end{aligned}$$

Each set of values of $u_i \in \{0,1\}$, $i = 0,\ldots,8$; $z3, z4, z5 = 1,\ldots,17$; $\lambda^{(j)}_5$ and $\bar{\lambda}^{(j)}_5$, $j = 6,7,8$ for which the equalities

$$L_{0,5} = L_{1,4} = L_{2,3} = L_{3,2} = L_{4,1} = L_{5,0} := \Lambda$$

is satisfied for a positive integer Λ will yield a simple 5-$(38,8,\Lambda)$ design. Recall that a 5-$(38,8,\Lambda)$ can be written as 5-$(38,8,m4)$ with $\lambda_{\min} = 4$ and $\lambda_{\max} = 5456$. Thus LIM $= \lfloor 5456/2 * 4 \rfloor = 682$. By solving the equalities above we obtain all solutions for $m \leq 1364$. Altogether 33 values for m have been found, of which 16 values of $m \leq$ LIM. Since, not all simple 5-$(19,i,\lambda^{(i)}_5)$ designs are known to exist, for example, 5-$(19,7,m7)$ designs are known for $m = 4,5,6,7,8,9,13$ only, we just obtain the following 5 new simple 5-$(38,8,m4)$ designs for $m = 280, 488, 524, 560, 560$ (the number 560 repeats twice, as we have two distinct non isomorphic solutions for this value of m). The details of these 5 constructed designs are given in Table 1.

Table 1: Constructed simple 5-$(38,8,\Lambda)$ designs

m	z_3	z_4	z_5	$\lambda^{(6)}_5$	$\lambda^{(7)}_5$	$\lambda^{(8)}_5$
280	7	0	7	0	35	56
488	8	4	8	4	28	280
524	6	7	6	6	28	196
560	4	10	4	8	28	112
560	9	5	9	4	49	112

An entry 0 in a column of the table implies that $u_i = 0$, otherwise $u_i = 1$. Here we have $\lambda^{(j)}_5 = \bar{\lambda}^{(j)}_5$, $j = 6,7,8$ for all these solutions.

4.1.2 Simple 5-$(38, k, \Lambda)$ designs with $k = 9, 10$

Again we assume that $v_1 = v_2 = 19$ for the construction of simple 5-$(38, k, \Lambda)$ designs with $k = 9, 10$.

- For construction of 5-$(38, 9, \Lambda)$ = 5-$(38, 9, m30)$ designs with LIM = 682, we make use of the large sets LS[17]$(2, i, 19)$, $i = 3, 4, 5, 6$, i.e. the 2-resolutions of the complete designs i-$(19, i, 1)$ with resolution class number $N_i = 17$. Thus, we have $R = \{(3, 6), (4, 5), (5, 4), (6, 3)\}$. And the equalities $L_{r,t-r}$ are the following.

$$\begin{aligned}
L_{0,5} &= u_0\bar{\lambda}_5^{(9)} + u_1 19\bar{\lambda}_5^{(8)} + u_2 171\bar{\lambda}_5^{(7)} + u_3 57 \times 14 z_3 + u_4 228 z_4,\\
L_{1,4} &= u_1 15\bar{\lambda}_5^{(8)}/4 + u_2 18 \times 5\bar{\lambda}_5^{(7)} + u_3 9 \times 105 z_3 + u_4 48 \times 15 z_4 + u_5 180 z_5,\\
L_{2,3} &= u_2 20\bar{\lambda}_5^{(7)} + u_3 560 z_3 + u_4 8 \times 120 z_4 + u_5 40 \times 16 z_5 + u_6 140 z_6,\\
L_{3,2} &= u_7 20\lambda_5^{(7)} + u_6 560 z_6 + u_5 120 \times 8 z_5 + u_4 16 \times 40 z_4 + u_3 140 z_3,\\
L_{4,1} &= u_8 15\lambda_5^{(8)}/4 + u_7 5 \times 18\lambda_5^{(7)} + u_6 105 \times 9 z_6 + u_5 15 \times 48 z_5 + u_4 180 z_4,\\
L_{5,0} &= u_9\lambda_5^{(9)} + u_8 19\lambda_5^{(8)} + u_7 171\lambda_5^{(7)} + u_6 14 \times 57 z_6 + u_5 228 z_5.
\end{aligned}$$

Solving the equalities $L_{0,5} = L_{1,4} = L_{2,3} = L_{3,2} = L_{4,1} = L_{5,0} = \Lambda$ for $\Lambda > 0$ with respect to $z_i = 1, \ldots, 17$ we obtain 20 values for m with $m \leq$ LIM leading to simple 5-$(38, 9, \Lambda)$ = 5-$(38, 9, m30)$ designs. Of which 14 designs can be constructed whose details are given in Table 2.

Table 2: Constructed simple 5-$(38, 9, \Lambda)$ designs

m	z_3	z_4	z_5	z_6	$\lambda_5^{(7)}$	$\lambda_5^{(8)}$	$\lambda_5^{(9)}$	$\bar{\lambda}_5^{(7)}$	$\bar{\lambda}_5^{(8)}$	$\bar{\lambda}_5^{(9)}$
100	2	1	1	2	0	56	112	0	56	112
200	4	2	2	4	0	112	224	0	112	224
300	6	3	3	6	0	168	336	0	168	336
400	8	4	4	8	0	224	448	0	224	448
402	5	5	5	5	28	84	546	28	84	546
500	10	5	5	10	0	280	560	0	280	560
502	7	6	6	7	28	140	658	28	140	658
504	4	7	7	4	56	0	756	56	0	756
582	10	4	11	3	28	168	588	63	84	189
602	9	7	7	9	28	196	770	28	196	770
604	6	8	8	6	56	56	868	56	56	868
660	9	8	8	9	35	252	21	35	252	21
680	8	11	4	15	0	364	602	35	280	203
682	5	12	5	12	28	224	700	63	140	301

It should be noted that when applying the basic construction for $t = 5$, $v_1 = v_2 = 19$ and $k = 8, 9$ we only obtain the trivial solutions, namely the complete

5-$(38, 8, 1364 \times 4)$ and 5-$(38, 9, 1364 \times 30)$ designs. This could be explained as follows.

In general, if $k \leq 2t-1$, then one of the designs in each pair $(D_i, \bar{D}_{k-i})$ is either the empty or the trivial design and at least one pair having both the trivial designs, therefore it leaves little room for the basic construction to produce a non-trivial solution, unless many pairs are unused, i.e. $u_i = 0$. The construction using resolutions indeed makes more room to create non-trivial solutions, as we have seen in the above examples.

- For construction of 5-$(38, 10, \Lambda)$ = 5-$(38, 10, m6)$ designs with LIM = 19778, we again employ the 2-resolutions of the complete designs i-$(19, i, 1)$ for $i = 3, 4, 5, 6, 7$ with resolution class number $N_i = 17$. Here,

$$R = \{(3,7), (4,6), (5,5), (6,4), (7,3)\}.$$

And we have

$$\begin{aligned}
L_{0,5} &= u_0\bar{\lambda}_5^{(10)} + u_1 19\bar{\lambda}_5^{(9)} + u_2 171\bar{\lambda}_5^{(8)} + u_3 57 \times 91 z_3 + u_4 228 \times 14 z_4 + u_5 684 z_5,\\
L_{1,4} &= u_1 3\bar{\lambda}_5^{(9)} + u_2 18 \times 15\bar{\lambda}_5^{(8)}/4 + u_3 9 \times 455 z_3 + u_4 48 \times 105 z_4 + u_5 180 \times 15 z_5\\
&\quad + u_6 504 z_6,\\
L_{2,3} &= u_2 12\bar{\lambda}_5^{(8)} + u_3 1820 z_3 + u_4 8 \times 560 z_4 + u_5 40 \times 120 z_5 + u_6 140 \times 16 z_6\\
&\quad + u_7 364 z_7,\\
L_{3,2} &= u_8 12\lambda_5^{(8)} + u_7 1820 z_7 + u_6 560 \times 8 z_6 + u_5 120 \times 40 z_5 + u_4 16 \times 140 z_4\\
&\quad + u_3 364 z_3,\\
L_{4,1} &= u_9 3\lambda_5^{(9)} + u_8 15 \times 18\lambda_5^{(8)}/4 + u_7 455 \times 9 z_7 + u_6 105 \times 48 z_6 + u_5 15 \times 180 z_5\\
&\quad + u_4 504 z_4,\\
L_{5,0} &= u_{10}\lambda_5^{(10)} + u_9 19\lambda_5^{(9)} + u_8 171\lambda_5^{(8)} + u_7 91 \times 57 z_7 + u_6 14 \times 228 z_6 + u_5 684 z_5.
\end{aligned}$$

Solving the equalities $L_{0,5} = L_{1,4} = L_{2,3} = L_{3,2} = L_{4,1} = L_{5,0} = \Lambda$ for $\Lambda > 0$ with respect to $z_i = 1, \ldots, 17$ we obtain an entire number of 479 solutions, of which 239 have $m \leq$ LIM. From these 239 parameters 131 simple 5-$(38, 10, m6)$ designs have been shown to exist. The values of m for these designs are

12768	17416	2604	6076	7252	10724	13668	15108
15372	18580	18844	3768	6976	8416	8680	11624
11888	12152	16272	16536	16800	19744	4932	8404
9580	9844	12788	13052	13316	13580	17172	17436
17700	17964	18228	6096	9040	10480	10744	11008
13952	11536	14216	14480	15920	18600	18864	19128
7260	11644	11908	12172	14852	15116	15380	15644
16556	16820	19500	17084	19764	8424	11368	12544
12808	13072	13336	16016	16280	16544	16808	17984
9060	9588	13972	14236	16916	14500	17180	17444
17708	18884	19148	10224	10752	14872	15136	15400
18080	15664	18344	18608	19520	11388	11916	16036
16300	16564	19244	16828	19508	19772	12552	13080
17200	17464	17728	13716	14244	18100	18364	18628
18892	19156	14880	15408	19264	19528	16044	17208
18384	18372	19536	16844	11316	13908	14280	14808
19720	16872	17772					

Here are two examples:

- 5-$(38, 10, 2604 \times 6)$ with $z_3 = 1$, $z_4 = 2$, $z_6 = 2$, $z_7 = 1$, $\bar{\lambda}_5^{(9)} = 147$, $\bar{\lambda}_5^{(10)} = 1260$, $u_2 = u_5 = u_8 = 0$ and $\lambda_5^{(i)} = \bar{\lambda}_5^{(i)}$ for $i = 9, 10$.
- 5-$(38, 10, 11316 \times 6)$ with $z_3 = 2$, $z_4 = 8$, $z_5 = 2$, $z_6 = 7$, $z_7 = 4$, $\bar{\lambda}_5^{(8)} = 140$, $\bar{\lambda}_5^{(9)} = 336$, $\bar{\lambda}_5^{(10)} = 294$, $\lambda_5^{(8)} = 84$, $\lambda_5^{(9)} = 378$, $\lambda_5^{(10)} = 1890$.

On the other hand, when the basic construction is applied for this case (i.e. $v_1 = v_2 = 19$ and $k = 10$), we just obtain 5 solutions with $m \leq \text{LIM}$.

Remark 4.1 1. It should be noted that when $v_1 = v_2$, any solution with $\lambda_t^{(i)} \neq \bar{\lambda}_t^{(i)}$ will appear twice by reason of symmetry, since $\lambda_t^{(i)}$ and $\bar{\lambda}_t^{(i)}$ may be interchanged. These two solutions are indeed the same. This fact should be taken into account by counting the number of solutions throughout Section 4.

2. Up to now the number of known parameter sets for 5-$(38, k, \Lambda)$ with $k = 8, 9, 10$ are 8, 14, and 23 respectively, see [12], for instance. For $k = 8, 9$ all the parameters of the constructed designs differ from the known ones. For $k = 10$, only one of the 23 known parameter sets does appear in the list of 131 constructed designs, namely the parameters 5-$(38, 10, 11368 \times 6)$. However, it is not known whether the corresponding designs are isomorphic.

4.2 Some further results of applications

We briefly record some further examples of simple t-designs for $t = 4, 5, 6$ by using Theorem 3.1.

4.2.1 $t = 4$

Following are several small parameters for $t = 4$.

1. 4-$(26, 8, m35)$: Take $v_1 = v_2 = 13$ and $R = \{(3,5),(4,4),(5,3)\}$ by using LS[55]$(2,4,13)$ and LS[11]$(2,i,13)$ for $i = 3,5$. There are 3 non-trivial solutions of Eq(3) with $m = 44, 66$ satisfying $m \le \text{LIM}(= 104)$. A design with $m = 44$ is known. The two solutions for $m = 66$ are non-isomorphic and new. These are

   - $u_4 = 0$, $z_3 = z_5 = 7$, $\lambda_4^{(7)} = 42$, $\lambda_4^{(8)} = 126$, $u_2 = u_6 = 0$, and $\bar{\lambda}_4^{(i)} = \lambda_4^{(i)}$ for $i = 7, 8$.
   - $z_4 = 24$, $z_3 = z_5 = 2$, $\lambda_4^{(6)} = 18$, $\lambda_4^{(8)} = 126$, $u_1 = u_7 = 0$, and $\bar{\lambda}_4^{(i)} = \lambda_4^{(i)}$ for $i = 6, 8$.

   The basic construction for 4-$(26, 8, m35)$ with $v_1 = v_2 = 13$ only yields the trivial solution.

2. 4-$(28, 9, m168)$: Take $v_1 = v_2 = 14$ and $R = \{(4,5),(5,4)\}$ by using LS[11]$(2,i,14)$ for $i = 4,5$. There is a unique non-trivial solution of Eq(3) with $m = 110$ satisfying $m \le \text{LIM}(= 126)$. This solution with $z_4 = z_5 = 4$, $u_2 = u_7 = 0$, $\lambda_4^{(6)} = 30$, $\lambda_4^{(8)} = 210$, $\lambda_4^{(9)} = 252$, and $\bar{\lambda}_4^{(i)} = \lambda_4^{(i)}$ for $i = 6, 8, 9$ yields a new design.

3. 4-$(30, 7, m20)$: Take $v_1 = v_2 = 15$ and $R = \{(3,4),(4,3)\}$ by using LS[13]$(2,i,15)$ for $i = 3,4$. There are 3 non-trivial solutions of Eq(3) with $m = 39, 52, 65$ satisfying $m \le \text{LIM}(= 65)$. The solution for $m = 52$ with $z_3 = z_5 = 5$, $\lambda_4^{(5)} = 5$, $\lambda_4^{(6)} = 15$, $\lambda_4^{(7)} = 115$, and $\bar{\lambda}_4^{(i)} = \lambda_4^{(i)}$ for $i = 5, 6, 7$ gives a new design.

4.2.2 $t = 5$

1. 5-$(36, 10, m63)$: Take $v_1 = v_2 = 18$ and $R = \{(5,5)\}$ by using LS[7]$(2,5,18)$. There are 164 non-trivial solutions of Eq(3) with $m \le \text{LIM}(= 1348)$. Of which 37 are shown to exist. It is interesting to remark that these 37 designs include the 10 designs constructed using the basic construction [39]. Actually, 27 new designs with parameters 5-$(36, 10, m63)$ have been obtained. These are

   $$m = 611, 818, 921, 945, 969, 1048, 1072, 911, 934, 1094, 1197, 1221, 1245, 1269, 1324, 1325, 1348, 1187, 1210, 1234, 1337, 1152, 1176, 1200, 1224, 1303, 1131.$$

2. 5-$(37, 8, m40)$: Take $v_1 = 13$, $v_2 = 24$ and $R = \{(3,5),(4,4),(5,3)\}$ by using LS[11]$(2,i,13)$, LS[11]$(2,i,24)$ for $i = 3,4,5$. There is a unique non-trivial solution of Eq(3) with $m = 55$ such that $m \le \text{LIM}(= 62)$. This solution with $z_3 = 2$,$z_5 = 2$, $z_4 = 8$, $\lambda_5^{(6)} = 4$, $\lambda_5^{(7)} = 28$, $\lambda_5^{(8)} = 56$, $\bar{\lambda}_5^{(6)} = 13$, $\bar{\lambda}_5^{(7)} = 36$, $\bar{\lambda}_5^{(7)} = 666$ gives a new design.

3. 5-$(37, 9, m10)$: Take $v_1 = 13$, $v_2 = 24$ and $R = \{(3,6), (4,5), (5,4), (6,3)\}$ by using LS[11]$(2, i, 13)$, LS[11]$(2, i, 24)$ for $i = 3, 4, 5, 6$. There is a unique non-trivial solution of Eq(3) with $m = 874$ such that $m \leq$ LIM$(= 1798)$. This solution with $z_3 = 2$, $z_4 = 2$, $z_5 = 4$, $z_6 = 1$, $\lambda_5^{(7)} = 14$, $u_8 = u_9 = 0$, $\bar{\lambda}_5^{(6)} = 72$, $\bar{\lambda}_5^{(7)} = 30$, $\bar{\lambda}_5^{(8)} = 1980$ gives a new design.

4. 5-$(44, 8, m)$: Take $v_1 = v_2 = 22$ and $R = \{(4,4)\}$ by using LS[19]$(2, 5, 22)$. There are 9 non-trivial solutions of Eq(3) with $m \leq$ LIM$(= 4569)$. Of which one design with $m = 3344$ and $u_3 = u_5 = 0$, $z_4 = 4$, $\lambda_5^{(6)} = 12$, $\lambda_5^{(7)} = 16$, $\lambda_5^{(8)} = 220$, and $\bar{\lambda}_5^{(i)} = \lambda_5^{(i)}$ for $i = 6, 7, 8$, is shown to exist.

5. 5-$(46, 10, m2)$: Take $v_1 = v_2 = 23$ and $R = \{(4,6), (5,5), (6,4)\}$ by using LS[133]$(2, 5, 23)$, LS[7]$(2, i, 23)$ for $i = 4, 6$. There are 3986 non-trivial solutions of Eq(3) with $m \leq$ LIM$(= 187349)$. Of which 176 designs are shown to exist with the following values of m.

65246	75487	73758	83999	86526	94240	96140	96767
106381	107008	116622	117021	125134	123405	127262	139004
137503	142633	139403	143887	149644	151772	159885	162013
164540	166668	174781	185497	59014	79667	78166	89908
88179	92435	98420	102676	108661	110561	120802	126160
122930	125058	131043	136401	133171	139555	137826	145540
153425	157054	158308	175807	174705	184319	182818	184946
77064	94088	99446	96216	104329	109687	102600	119928
112841	116698	121828	115368	123082	132069	124982	125609
130967	142310	135223	140581	135850	141208	145464	150822
145863	153976	156104	163590	167846	166345	171475	168245
178486	183844	180614	185972	86526	105355	113867	112366
118750	124108	126635	134349	127262	136249	134748	131518
136876	137503	146490	139403	147117	140030	156731	149644
155002	151772	157130	150271	153900	159885	165243	162013
167371	166668	177612	174382	179512	182267	185896	187150
121505	119776	128288	132544	138529	142785	148770	145540
150670	160911	163039	159809	171152	169423	164692	170050
168321	179664	174705	182818	184946	135850	148694	152950
158707	163191	165091	163590	165718	171076	175332	179189
175959	181317	174230	185573	178486	183844	180614	185972
184471	182742	176985	164540	177612	179512	187226	173052

Here is an example with $m = 59014$: $z_4 = z_6 = 1$, $z_5 = 20$, $u_2 = u_8 = 0$, $\lambda_5^{(7)} = 36$, $\lambda_5^{(9)} = 810$, $\lambda_5^{(10)} = 7812$, and $\bar{\lambda}_5^{(i)} = \lambda_5^{(i)}$ for $i = 7, 9, 10$.

4.2.3 $t = 6$

Following are some examples for $t = 6$.

1. 6-$(38, 10, m10)$: Take $v_1 = v_2 = 19$ and $R = \{(4, 6), (5, 5), (6, 4)\}$ by using LS[4]$(3, i, 19)$ for $i = 4, 5, 6$. There are 4 non-trivial solutions of Eq(3) with $m = 1360, 892, 1340, 1788$ for $m \leq \text{LIM}(= 1798)$.

2. 6-$(46, 12, m420)$: Take $v_1 = v_2 = 23$ and $R = \{(6, 6)\}$ by using LS[3]$(3, 6, 23)$. There are 2 non-trivial solutions of Eq(3) with $m = 3363, 3819$ for $m \leq \text{LIM}(= 4569)$. The solution for $m = 3363$ has $z_6 = 1$, $\lambda_5^{(7)} = 7$, $\lambda_5^{(8)} = 40$, $\lambda_5^{(9)} = 340$, $\lambda_5^{(10)} = 350$, $\lambda_5^{(11)} = 4046$, $\lambda_5^{(12)} = 5320$, and $\bar{\lambda}_5^{(i)} = \lambda_5^{(i)}$ for $i = 7, 8, 9, 10, 11, 12$. All the ingredient designs corresponding to $m = 3363$ exist except that the existence of a 6-$(23, 10, 5 \times 70)$ design is still in doubt. So, we would have a 6-$(46, 12, 3364 \times 420)$ design if a 6-$(23, 10, 5 \times 70)$ design would exist.

3. 6-$(50, 12, m308)$: Take $v_1 = v_2 = 25$ and $R = \{(6, 6)\}$ by using LS[7]$(3, 6, 25)$. There are 195 non-trivial solutions of Eq(3) for $m \leq \text{LIM}(= 11459)$.

5 Conclusion

We have presented a recursive construction for simple t-designs by using the concept of resolutions. This may be viewed as an extension of the basic construction as shown in our previous paper. The s-resolutions of trivial t-designs are equivalent to the large sets of s-designs, which have been extensively studied. Since our construction does not exclude the use of trivial designs as ingredients, we have restricted its applications to resolutions of the trivial ingredient designs only. In spite of this fact, the construction still produces a large number of new simple t-designs. We strongly believe that the construction would unfold its full impact when we would gain more knowledge about resolutions of non-trivial t-designs.

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