NOTES ON FETI-DP DOMAIN DECOMPOSITION METHODS FOR P-ELASTICITY

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1. Introduction. This technical report is meant to be read as a companion paper to [7]. Here, we present some details and proofs not given in [7].

The paper [7] deals with the solution of the elasticity model

$$\int_\Omega 2\mu_e \langle \operatorname{sym}(P^{-T}\nabla u), \operatorname{sym}(P^{-T}\nabla v)\rangle + \lambda_e \operatorname{tr}(P^{-T}\nabla u)\operatorname{tr}(P^{-T}\nabla v)\, d\boldsymbol{x} = (F, v)_{L_2(\Omega)}, \quad (1.1)$$

using a FETI-DP domain decomposition method. Here, P is a 3×3-matrix which is a new parameter introduced to obtain a broader range of application for this model than for standard linear elasticity.

This technical report is organized as follows. In Section 2 we present the detailed concept of faces, edges, and vertices as used in the FETI-DP methods. In Section 3 we discuss the existence of a sufficient number of primal constraints to control the kernel of the stiffness matrices. Generalized Korn inequalities are presented and proven in Section 4 since they are needed to obtain ellipticity. In Section 5, we present the auxiliary lemmas needed for the analysis in [7] together with proofs as far as the proofs are not given in the appendix of [7].

2. Faces, edges, and vertices. Here, we follow the presentation given in Klawonn and Rheinbach [8, Section 2]; see also Klawonn and Widlund [9]. We denote individual faces, edges, and vertices by $\mathcal{F}, \mathcal{E}$, and $\mathcal{V}$, respectively. To define faces, edges, and vertices, we introduce certain equivalence classes. Let us denote the sets of nodes on $\partial\Omega, \partial\Omega_i$, and Γ by $\partial\Omega_h, \partial\Omega_{i,h}$, and Γ_h, respectively. For any interface nodal point $x \in \Gamma_h$, we define

$$\mathcal{N}_x := \{j \in \{1,\ldots,N\} : x \in \partial\Omega_{j,h}\},$$

i.e., $\mathcal{N}_x$ is the set of indices of all subdomains with x in the closure of the subdomain. For a node x we define the multiplicity as $|\mathcal{N}_x|$.

Associated with the nodes of the finite element mesh, we have a graph, the nodal graph, which represents the node-to-node adjacency. For a given node $x \in \Gamma_h$, we denote by $\mathcal{C}_{\text{con}}(x)$ the connected component of the nodal subgraph, defined by $\mathcal{N}_x$, to which x belongs. For two interface points $x, y \in \Gamma_h$, we introduce an equivalence relation by

$$x \sim y :\Leftrightarrow \mathcal{N}_x = \mathcal{N}_y \quad \text{and} \quad y \in \mathcal{C}_{\text{con}}(x).$$

We can now describe faces, edges and vertices using their equivalence classes. Here, $|G|$ denotes the cardinality of the set G. We define the following.

Definition 1.

$$x \in \mathcal{F} :\Leftrightarrow |\mathcal{N}_x| = 2.$$
$$x \in \mathcal{E} :\Leftrightarrow |\mathcal{N}_x| \geq 3 \;\text{ and }\; \exists y \in \Gamma_h, y \neq x, \;\text{ such that }\; y \sim x.$$
$$x \in \mathcal{V} :\Leftrightarrow |\mathcal{N}_x| \geq 3 \;\text{ and }\; \nexists y \in \Gamma_h, y \neq x, \;\text{ such that }\; y \sim x.$$

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In the case of a decomposition into regular substructures, e.g., cubes or tetrahedra, our definition of faces, edges, and vertices conforms to our basic geometric intuition. On the other hand, for subdomains generated by an automatic mesh partitioner, the situation can be quite complicated. We can, e.g., have several edges with the same index set $\mathcal{N}_x$ or an edge and a vertex with the same $\mathcal{N}_x$. In practice, we can also have situations when there are not enough edges and potential edge constraints for some subdomains. Then we have to use constraints on some extra edges on $\partial\Omega_N$, which otherwise would be regarded as part of a face. A similar problem might occure for flat structures for which additional constraints might be required for each subdomain. Therefore, we introduce an alternative definition of edges.

DEFINITION 2. *An edge is the largest connected set of nodes with the same index set $\mathcal{N}_x$, where $\mathcal{N}_x \geq 3$ or $\mathcal{N}_x \geq 2$ and x is on $\partial\Omega_N$.*

If needed, we can increase the number of edges in unstructured cases by switching locally from definition of edges given in Definition 1 to Definition 2 and by splitting edges into several edges.

3. Primal constraints. To ensure that the local stiffness matrices are invertible we need to control the kernel of the bilinear form (1.1). Hence, we need to show that there exist six primal constraints, which control the kernel and are linear independent, i.e, we need to prove the following Lemma 1.

LEMMA 1. *Let $P^{-T} = \nabla\psi$ and ψ be a $\mathcal{C}^1$-diffeomorphism with $\det(\nabla\psi)$ being bounded from below and above, i.e., $0 < c \leq |\det(\nabla\psi)| \leq C < \infty$ Then, for every subdomain face and for the standard case, cf.[7, Section 7, Assumption 1], we can always find six edge averages of the displacement components that are linearly independent when restricted to the space $\mathbf{ker}\,(\varepsilon_P)$.*

Proof: First we will consider the elements $\mathbf{r}_4, \mathbf{r}_5$, and $\mathbf{r}_6$ of $\mathbf{ker}\,(\varepsilon_P)$, i.e.,

$$\mathbf{r}_4(\mathbf{x}) := \frac{1}{H_\psi}\begin{bmatrix} \psi^{(2)}(\mathbf{x}) - \psi^{(2)}(\hat{\mathbf{x}}) \\ -\psi^{(1)}(\mathbf{x}) + \psi^{(1)}(\hat{\mathbf{x}}) \\ 0 \end{bmatrix} \quad , \quad \mathbf{r}_5(\mathbf{x}) := \frac{1}{H_\psi}\begin{bmatrix} -\psi^{(3)}(\mathbf{x}) + \psi^{(3)}(\hat{\mathbf{x}}) \\ 0 \\ \psi^{(1)}(\mathbf{x}) - \psi^{(1)}(\hat{\mathbf{x}}) \end{bmatrix},$$

$$\mathbf{r}_6(\mathbf{x}) := \frac{1}{H_\psi}\begin{bmatrix} 0 \\ \psi^{(3)}(\mathbf{x}) - \psi^{(3)}(\hat{\mathbf{x}}) \\ -\psi^{(2)}(\mathbf{x}) + \psi^{(2)}(\hat{\mathbf{x}}) \end{bmatrix}, \tag{3.1}$$

cf. [7, Section 3, (15)]. For $\mathbf{w} = (w^{(j)})_{j=1,2,3}$ we consider

$$g(\mathbf{w}) = \frac{\int_{\mathcal{E}^{ik}} w^{(j)}(\mathbf{x})\ d\mathbf{x}}{\int_{\mathcal{E}^{ik}} 1\ d\mathbf{x}}.$$

Since we want to control the basis elements of $\mathbf{ker}\,(\varepsilon_P)$ we have to evaluate g for these elements

$$g(\mathbf{r}_n) = \frac{\int_{\mathcal{E}^{ik}} r_n^{(j)}(\mathbf{x})\ d\mathbf{x}}{\int_{\mathcal{E}^{ik}} 1\ d\mathbf{x}} \quad \text{for } n = 4, 5, 6.$$

Because ψ is a C^1-diffeomorphism, we can carry out a change of variables

$$\psi : \Omega_i \to \widehat{\Omega}_i \quad , \quad \mathbf{x} \mapsto \xi := \psi(\mathbf{x}).$$

By using the transformation formula, we obtain

$$g(\mathbf{r}_n) = \frac{\int_{\xi\in\psi(\mathcal{E}^{ik})} \mathbf{r}_n^{(j)}(\psi^{-1}(\xi))\,|\det(\nabla\psi^{-1}(\xi))|\;d\xi}{\int_{\xi\in\psi(\mathcal{E}^{ik})} |\det(\nabla\psi^{-1}(\xi))|\;d\xi}$$

and by using the special form of $\mathbf{r}$ in (3.1) we have

$$\mathbf{r}_4(\psi^{-1}(\xi)) = \begin{bmatrix} \psi^{(2)}(\psi^{-1}(\xi)) - \psi^{(2)}(\psi^{-1}(\hat\xi)) \\ -\psi^{(1)}(\psi^{-1}(\xi)) + \psi^{(1)}(\psi^{-1}(\hat\xi)) \\ 0 \end{bmatrix} = \begin{bmatrix} \xi_2 - \hat\xi_2 \\ -\xi_1 + \hat\xi_1 \\ 0 \end{bmatrix} =: \tilde{\mathbf{r}}_4(\xi).$$

For $n = 5, 6$, we obtain analogously

$$\tilde{\mathbf{r}}_5(\xi) := \begin{bmatrix} -\xi_3 + \hat\xi_3 \\ 0 \\ \xi_1 - \hat\xi_1 \end{bmatrix} \quad , \quad \tilde{\mathbf{r}}_6(\xi) := \begin{bmatrix} 0 \\ \xi_3 - \hat\xi_3 \\ -\xi_2 + \hat\xi_2 \end{bmatrix}.$$

Such we have for $n = 4, 5, 6$

$$g(\mathbf{r}_n) = \frac{\int_{\xi\in\psi(\mathcal{E}^{ik})} \tilde{\mathbf{r}}_n^{(j)}(\xi)\,|\det(\nabla\psi^{-1}(\xi))|\;d\xi}{\int_{\xi\in\psi(\mathcal{E}^{ik})} |\det(\nabla\psi^{-1}(\xi))|\;d\xi}.$$

Since the entries in $\mathbf{r}_n$ are constant for $n = 1, 2, 3$, because they are the same translation vectors are in standard linear elasticity, it is obvious that we obtain

$$\mathbf{r}_n(\mathbf{x}) = \tilde{\mathbf{r}}_n(\xi) \quad n = 1, 2, 3.$$

The functions $\tilde{\mathbf{r}}_n$, $n = 1, \ldots, 6$, have the form of the standard basis of the space of rigid body modes from linear elasticity. Since we have assumed that the determinant of P^{-T} is bounded from below and above we obtain

$$\frac{c}{C}\;\underbrace{\frac{\int_{\xi\in\psi(\mathcal{E}^{ik})} \tilde{\mathbf{r}}_n^{(j)}(\xi)\;d\xi}{\int_{\xi\in\psi(\mathcal{E}^{ik})} 1\;d\xi}}_{=:\tilde g(\tilde{\mathbf{r}}_n)} \;\le\; g(\mathbf{r}_n) \;\le\; \frac{C}{c}\;\underbrace{\frac{\int_{\xi\in\psi(\mathcal{E}^{ik})} \tilde{\mathbf{r}}_n^{(j)}(\xi)\;d\xi}{\int_{\xi\in\psi(\mathcal{E}^{ik})} 1\;d\xi}}_{=\tilde g(\tilde{\mathbf{r}}_n)}. \tag{3.2}$$

It was shown by Klawonn and Widlund [9, Proposition 5.1], that the lemma holds for rigid body modes $\tilde{\mathbf{r}}_n$ and the functionals $\tilde g$.

From (3.2) also follows that six linear independent functionals g_n exist. Let $g_n(\mathbf{r}) = 0\ \forall n = 1, \ldots 6$ then (3.2) implies that $\tilde g_n(\tilde{\mathbf{r}}) = 0\ \forall n = 1, \ldots, 6$. But since the lemma is true for the $\tilde g_n$ it follows that $\tilde{\mathbf{r}} = 0$. Because the transformation only affects the basis vectors but not the coefficients it follows that $\mathbf{r} = 0$ and hence the lemma also holds for the case of P-elasticity when P^{-T} is a gradient. ☐

4. Korn inequalities. To ensure unique solvability of our problem we need ellipticity of the bilinear form in (1.1). Therefore, we need two generalized Korn inequalities. These inequalities can be found in [10]. Here, we repeat them with detailed proofs since we are interested in the dependence of the constants c^+ on the parameter P.

Theorem 1. *(Generalized Korn's first inequality, [10, Theorem 4.13])* *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $\Sigma \subset \partial\Omega$ be a smooth part of the boundary with nonvanishing two-dimensional Lebesgue measure. Let*

$$\mathbf{H}_0^1(\Omega, \Gamma) := \{\phi \in \mathbf{H}^1(\Omega) \mid \phi_{|\Gamma} = 0\}$$

and let $P^{-T} = \nabla\psi \in C^0(\bar{\Omega}, \mathbb{R}^{3\times 3}) \subset L^\infty(\bar{\Omega}, \mathbb{R}^{3\times 3})$ *be given with a positive constant* a^+ *such that* $\det P^T \geq a^+$ *and let* $\psi : \bar{\Omega} \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$ *be a* C^1*-diffeomorphism. Then there exists a constant* $c^+ > 0$ *such that*

$$\|(\nabla\phi)P^T(\mathbf{x}) + P(\mathbf{x})(\nabla\phi)^T\|^2_{L_2(\Omega)} \geq c^+\|\phi\|^2_{H^1(\Omega)} \quad \forall\phi \in \mathbf{H_0^1}(\Omega, \Gamma).$$

Proof: The proof given here can be found mainly in [10]; for the convenience of the reader, it is repeated here using our notation and working out the dependence of the constants on P.

Since ψ is assumed to be a diffeomorphism, we interprete it as a transformation of variables and define $\xi := \psi(\mathbf{x})$, cf. Section 3.

As $C_0^\infty(\Omega, \Gamma)$ is dense in $\mathbf{H_0^1}(\Omega, \Gamma)$, we can assume that $\phi \in C_0^\infty(\Omega, \Gamma)$ and obtain the estimate for $\mathbf{H_0^1}(\Omega, \Gamma)$ by density. With ϕ we construct another function ϕ_e

$$\phi_e(\psi(\mathbf{x})) = \phi_e(\xi) := \phi(\psi^{-1}(\xi)) = \phi(\psi^{-1}(\psi(\mathbf{x}))) = \phi(\mathbf{x}).$$

This function ϕ_e is differentiable with a gradient

$$\nabla_{\mathbf{x}}\phi(\mathbf{x}) = \nabla_{\mathbf{x}}(\phi_e(\xi)) = (\nabla_\xi\phi_e(\xi))(\nabla_{\mathbf{x}}\psi(\mathbf{x}))$$

$$\Leftrightarrow (\nabla_{\mathbf{x}}\phi(\mathbf{x}))(\nabla_{\mathbf{x}}\psi(\mathbf{x}))^{-1} = \nabla_\xi\phi_e(\xi) = (\nabla_{\mathbf{x}}\phi(\mathbf{x}))P^T \tag{4.1}$$

$$\Leftrightarrow (\nabla_{\mathbf{x}}\phi(\mathbf{x}))\left(\nabla_\xi\psi^{-1}(\xi)\right) = \nabla_\xi\phi_e(\xi).$$

Instead of the given L_2-norm, we consider the expression in terms of ϕ_e and use the standard Korn's first inequality on the transformed domain $\psi(\Omega)$, cf. Ciarlet [4], [9, Lemma 2.1]. Note that the constant depends on $\psi(\Omega)$ and on $\psi(\Gamma) \subset \psi(\partial\Omega)$, i.e., $C := C(\psi(\Omega), \psi(\Gamma))$.

$$\int_{\xi\in\psi(\Omega)} \|\nabla_\xi\phi_e(\xi) + (\nabla_\xi\phi_e(\xi))^T\|^2 \, d\xi \geq C\int_{\xi\in\psi(\Omega)} \|\nabla_\xi\phi_e(\xi)\|^2 \, d\xi. \tag{4.2}$$

With the transformation formula we obtain for (4.2)

$$\int_\Omega \|\nabla_\xi\phi_e(\psi(\mathbf{x})) + (\nabla_\xi\phi_e(\psi(\mathbf{x})))^T\|^2 \, |\det(\nabla\psi(\mathbf{x}))| \, d\mathbf{x}$$
$$\geq C\int_\Omega \|\nabla\phi_e(\psi(\mathbf{x}))\|^2 \, |\det(\nabla\psi(\mathbf{x}))| \, d\mathbf{x}.$$

Since we have

$$1 = \det(\mathrm{Id}) = \det((\nabla\psi(\mathbf{x}))\cdot(\nabla\psi(\mathbf{x}))^{-1}) = \det(\nabla\psi(\mathbf{x}))\cdot\det(P^T)$$

$$\Leftrightarrow \quad 0 \leq \frac{1}{\det(P^T)} = \det((\nabla\psi(\mathbf{x})) \leq \frac{1}{a^+},$$

we can estimate $\det(\nabla\psi(\mathbf{x}))$ by its maximum over all $\mathbf{x} \in \Omega$ on the left hand side and by its minimum on the right hand side. By division we obtain

$$\|\nabla_\xi\phi_e(\psi(\mathbf{x})) + (\nabla_\xi\phi_e(\psi(\mathbf{x})))^T\|^2_{L_2(\Omega)} \geq C\,\frac{\min_{\mathbf{x}\in\Omega}\det(\nabla\psi(\mathbf{x}))}{\max_{\mathbf{x}\in\Omega}\det(\nabla\psi(\mathbf{x}))}\|\nabla\phi_e(\psi(\mathbf{x}))\|^2_{L_2(\Omega)}.$$

By using (4.1) we obtain

$$\|\nabla_{\mathbf{x}}\phi(\mathbf{x})P^T + P(\nabla_{\mathbf{x}}\phi(\mathbf{x}))^T\|^2_{L_2(\Omega)} \geq C\frac{\min_{\mathbf{x}\in\Omega}\det(\nabla\psi(\mathbf{x}))}{\max_{\mathbf{x}\in\Omega}\det(\nabla\psi(\mathbf{x}))}\|\nabla_{\mathbf{x}}\phi(\mathbf{x})P^T\|^2_{L_2(\Omega)}. \quad (4.3)$$

As we need an upper estimate for $\|\phi\|_{H^1(\Omega)}$, we have to examine $\|\nabla_{\mathbf{x}}\phi(\mathbf{x})P^T\|_{L_2(\Omega)}$ more closely.

$$\|(\nabla\phi)P^T(\mathbf{x})\|^2_{L_2(\Omega)} = \int_\Omega \operatorname{tr}\big(\underbrace{(\nabla\phi)}_{:=L}\underbrace{(P^T(\mathbf{x})P(\mathbf{x}))}_{:=N}(\nabla\phi)^T\big)\, d\mathbf{x}$$

$$= \int_\Omega \sum_{k=1}^3 \Big(\sum_{i,j=1}^3 l_{ki}n_{ij}l_{kj}\Big)\, d\mathbf{x} \quad =: B.$$

With l_k being the k-th row of L, we have, since N is symmetric,

$$\sum_{i,j=1}^3 l_{ki}n_{ij}l_{kj} \; = \; l_k N l_k^T \; = \; \langle N l_k^T, l_k^T\rangle.$$

We use a Rayleigh quotient argument for the smallest eigenvalue of N and obtain

$$\lambda_{\min}(N) \; = \; \min_{\substack{\mathbf{x}\in\mathrm{I\!R}^3\\ \mathbf{x}\neq 0}} \frac{\langle N\mathbf{x},\mathbf{x}\rangle}{\langle \mathbf{x},\mathbf{x}\rangle} \; \leq \; \frac{\langle N l_k^T, l_k^T\rangle}{\langle l_k^T, l_k^T\rangle}$$

$$\Rightarrow \lambda_{\min}(N)\langle l_k^T, l_k^T\rangle \; = \; \lambda_{\min}(N)\Big(\sum_{i=1}^3 l_{ki}^2\Big) \; \leq \; \langle N l_k^T, l_k^T\rangle.$$

To obtain a constant which is independent of $\mathbf{x}$, we define $\lambda_{\min,\Omega}(N)$ as $\inf_{\mathbf{x}\in\bar\Omega}(\lambda_{\min}(N))(\mathbf{x})$. This leads to

$$B \;\geq\; \lambda_{\min,\Omega}(P^TP)\int_\Omega \sum_{k=1}^3(\sum_{i=1}^3(\partial_k\phi_i)^2)\, d\mathbf{x} \;=\; \lambda_{\min,\Omega}(P^TP)\|\nabla\phi\|^2_{L_2(\Omega)}\,.$$

We combine this result with (4.3) and obtain

$$\|\nabla_{\mathbf{x}}\phi(\mathbf{x})P^T + P(\nabla_{\mathbf{x}}\phi(\mathbf{x}))^T\|^2_{L_2(\Omega)} \geq C\frac{\min_{\mathbf{x}\in\Omega}\det(P^{-T}(\mathbf{x}))}{\max_{\mathbf{x}\in\Omega}\det(P^{-T}(\mathbf{x}))}\lambda_{\min,\Omega}(P^TP)|\phi|^2_{H^1(\Omega)}.$$

Since Ω is a bounded Lipschitz domain and we have Dirichlet boundary conditions we can use a standard Poincaré-Friedrichs inequality. The desired inequality follows by a density argument. □

THEOREM 2. *(Korn's second inequality, [10, Corollary 4.7]) Let us consider the same assumptions as in Theorem 1. Then, there exists a constant* $c^+ > 0$ *such that*

$$\|(\nabla\phi)P^T(\mathbf{x}) + P(\mathbf{x})(\nabla\phi)^T\|^2_{L_2(\Omega)} + \|\phi\|^2_{L_2(\Omega)} \geq c^+\|\phi\|^2_{H^1(\Omega)} \quad \forall\phi\in\mathbf{H^1}(\Omega).$$

Proof: We can proceed in nearly the same way as in the proof of Theorem 1.

Since $C^\infty(\bar\Omega)$ is dense in $H^1(\Omega)$, we choose $\phi\in C^\infty(\bar\Omega)$. Then, we can complete our proof with a standard density argument. The function ϕ_e may also be defined as before. Hence we can also adopt the considerations concerning ϕ_e. Here we will use the standard second Korn inequality on the transformed domain $\psi(\Omega)$, cf. Nitsche [11], and obtain

$$\int_{\xi\in\psi(\Omega)} \|\nabla_\xi \phi_e(\xi) + (\nabla_\xi \phi_e(\xi))^T\|^2 \; d\xi + \|\phi_e\|^2_{L_2(\psi(\Omega))} \ge c(\psi(\Omega)) \|\phi_e\|^2_{H^1(\psi(\Omega))},$$

which can also be written in the following way

$$\int_{\xi\in\psi(\Omega)} \|\nabla_\xi \phi_e(\xi) + (\nabla_\xi \phi_e(\xi))^T\|^2 \; d\xi + \int_{\xi\in\psi(\Omega)} \|\phi_e(\xi)\|^2 \; d\xi \ge c \int_{\xi\in\psi(\Omega)} \|\nabla_\xi \phi_e(\xi)\|^2 + \|\phi_e(\xi)\|^2 \; d\xi\,,$$

where now a constant $c := c(\psi(\Omega))$ occurs, depending on the shape of the transformed domain. We use the transformation formula of integrals and estimate the determinant as before to obtain

$$\begin{aligned}
& \int_\Omega \|(\nabla_{\mathbf{x}}\phi) P^T(\mathbf{x}) + P(\mathbf{x})(\nabla_{\mathbf{x}}\phi)^T\|^2 + \|\phi\|^2 \; d\mathbf{x} \\
\ge \quad & c(\psi(\Omega)) \frac{\min_{\mathbf{x}\in\Omega} \det(\nabla\psi(\mathbf{x}))}{\max_{\mathbf{x}\in\Omega} \det(\nabla\psi(\mathbf{x}))} \int_\Omega \|(\nabla_{\mathbf{x}}\phi) P^T\|^2 + \|\phi\|^2 \; d\mathbf{x} \\
\ge \quad & c(\psi(\Omega)) \frac{\min_{\mathbf{x}\in\Omega} \det(\nabla\psi(\mathbf{x}))}{\max_{\mathbf{x}\in\Omega} \det(\nabla\psi(\mathbf{x}))} \min\{\lambda_{\min,\Omega}(P^TP), 1\} \left(|\phi|^2_{H^1(\Omega)} + \|\phi\|^2_{L_2(\Omega)}\right) \\
= \quad & c(\psi(\Omega)) \frac{\min_{\mathbf{x}\in\Omega} \det(P^{-T}(\mathbf{x}))}{\max_{\mathbf{x}\in\Omega} \det(P^{-T}(\mathbf{x}))} \min\{\lambda_{\min,\Omega}(P^TP), 1\} \; \|\phi\|^2_{H^1(\Omega)}. \qquad \square
\end{aligned}$$

5. Auxiliary lemmas with proofs. Here, we give some of the technical lemmas needed in [7, Section 7] with their proofs.

We need a Sobolev-type inequality for piecewise quadratic finite element functions.

LEMMA 2. *(see Lemma A.3 in [7]) Let $\mathcal{E}^{ik}$ be any edge of Ω_i that forms a part of the boundary of a face $\mathcal{F}^{ij} \subset \partial\Omega_i$. Then for all $\mathbf{u} \in \mathbf{W^{(i)}}$,*

$$\|\mathbf{u}\|^2_{L_2(\mathcal{E}^{ik})} \le C \left(1 + \log\left(\frac{H_i}{h_i}\right)\right) \left(|\mathbf{u}|^2_{H^{1/2}(\partial\Omega_i)} + \frac{1}{H_i}\|\mathbf{u}\|^2_{L_2(\partial\Omega_i)}\right).$$

Proof: For simplicity, we assume for the rest of the proof that u is a scalar finite element function. The result immediately carries over to the vector valued case by applying it component-by-component. To prove this lemma we first need a discrete Sobolev inequality in two dimensions. This estimate can be found in [3, Lemma (4.9.1)] for $\mathcal{P}_m$ Lagrange finite element functions. From [3, Lemma (4.9.1)], we have for a domain $\tilde{\Omega} \subset \mathbb{R}^2$ with $\operatorname{diam}(\tilde{\Omega}) = H$

$$\|u\|^2_{L^\infty(\tilde{\Omega})} \le C \left(1 + \log\left(\frac{H}{h}\right)\right) \|u\|^2_{H^1(\tilde{\Omega})},$$

for all $u \in \{v \in H^1(\tilde{\Omega}) : v \text{ piecewise in } \mathcal{P}_m\}$. With this estimate we can follow the line of arguments given in [12, Lemma 4.16], Bramble, Pasciak, and Schatz [1], and Bramble and Xu [2]. For convenience we assume that our edge $\mathcal{E}^{ik}$ is a straight line. Hence we can assume that $\mathcal{E}^{ik}$ can be described as $\{\mathbf{x} = (x, y, z) \in \mathbb{R}^3 : x \in I \wedge y = f(x) \wedge z = g(x)\}$ with a real open interval I and linear functions f and g each mapping from $\mathbb{R}$ to $\mathbb{R}$. With this parametrization we have

$$\|u\|^2_{L_2(\mathcal{E}^{ik})} = \int_I |u(x, f(x), g(x))|^2 \; dx.$$

Hence, we can estimate $|u(x, f(x), g(x))|$ by its maximum over a two dimensional cross section of Ω_i denoted as $\Omega_{i,x}$ associated with a point $(x, f(x), g(x))$ for each x, and obtain

$$\|u\|^2_{L_2(\mathcal{E}^{ik})} \leq \int_I \|u\|^2_{L^\infty(\Omega_{i,x})}\, dx \leq \int_I \left(C \left(1 + \log\left(\frac{H_i}{h_i}\right)\right) \|u\|^2_{H^1(\Omega_{i,x})}\right)\, dx.$$

And since the integral over I combined with the integral over $\Omega_{i,x}$ leads to an integral over Ω_i we have

$$\|u\|^2_{L_2(\mathcal{E}^{ik})} \leq C\left(1 + \log\left(\frac{H_i}{h_i}\right)\right) \|u\|^2_{H^1(\Omega_i)}.$$

This argument holds for any function with the same trace and therefore, for the harmonic extension $\mathcal{H}u$ we obtain

$$\|u\|^2_{L_2(\mathcal{E}^{ik})} \leq C\left(1 + \log\left(\frac{H_i}{h_i}\right)\right) \|\mathcal{H}u\|^2_{H^1(\Omega_i)}$$

and we conclude by using

$$|u|_{H^{1/2}(\partial\Omega)} := \inf_{\substack{v\in H^1(\Omega)\\ v|_{\partial\Omega}=u}} |v|_{H^1(\Omega)} \quad \text{for} \quad u \in H^{1/2}(\partial\Omega),$$

$$\text{and} \quad |\mathbf{u}|^2_{H^{1/2}(\partial\Omega)} := \sum_{i=1}^{3} |u_i|^2_{H^{1/2}(\partial\Omega)} \quad \text{for} \quad \mathbf{u} \in \mathbf{H^{1/2}}(\partial\Omega).$$

and the fact that the harmonic extension has the least energy. □

The next lemma can be found in the monograph by Toselli and Widlund [12, Lemma 4.28] for the case of piecewise linear finite element functions.

LEMMA 3. *(see Lemma A.4 in [7]) Let $\mathcal{V}^{il}$ be a vertex of a subdomain Ω_i and let $\mathbf{u} \in \mathbf{W^{(i)}}$. Then*

$$|\mathbf{u}(\mathcal{V}^{il})\theta_{\mathcal{V}^{il}}|^2_{H^{1/2}(\partial\Omega_i)} \leq C\left(|\mathbf{u}|^2_{H^{1/2}(\partial\Omega_i)} + \frac{1}{H_i}\|\mathbf{u}\|^2_{L_2(\partial\Omega_i)}\right).$$

Proof: As in the proof of the previous lemma, we assume without restrictions that u is a scalar finite element function. From [12, (4.16)] we obtain

$$\|u\|^2_{L^\infty(T)} \leq c\frac{1}{h}\|u\|^2_{H^1(T)}.$$

Using this estimate, we obtain

$$\begin{aligned} |u(\mathcal{V}^{il})\theta_{\mathcal{V}^{il}}|^2_{H^{1/2}(\partial\Omega_i)} &\leq |u(\mathcal{V}^{il})\theta_{\mathcal{V}^{il}}|^2_{H^1(\Omega_i)} \leq |u(\mathcal{V}^{il})|^2 |\theta_{\mathcal{V}^{il}}|^2_{H^1(\Omega_i)} \\ &= \sum_{T\subset\overline{\Omega}_i} |u(\mathcal{V}^{il})|^2 |\theta_{\mathcal{V}^{il}}|^2_{H^1(T)} \leq \sum_{T\subset\overline{\Omega}_i} c\frac{1}{h}\|u\|^2_{H^1(T)} |\theta_{\mathcal{V}^{il}}|^2_{H^1(T)}. \end{aligned}$$

It remains to estimate $|\theta_{\mathcal{V}^{il}}|^2_{H^1(\Omega_i)}$. The function $\theta_{\mathcal{V}^{il}}$ is linear and takes the value 1 in $\mathcal{V}^{il}$ and 0 in every other node. Its support is bounded by the volume of a tetrahedron and its gradient can be bounded by $\frac{2}{h}$. Hence, we obtain

$$|\theta_{\mathcal{V}^{il}}|^2_{H^1(T)} \leq c\frac{1}{h^2}h^3 = ch.$$ □

The following result can be found in Dryja, Smith, and Widlund [6, Lemma 4.5], Dryja [5, Lemma 3], and Toselli and Widlund [12, Lemma 4.24] but only for piecewise linear functions. Here, we present a version for piecewise quadratic finite element functions. For this case, it can be proven by combining the arguments given in the proof of [12, Lemma 4.24] with the same element by element techniques as applied for the previous lemmas of this section.

LEMMA 4. *(see Lemma A.3 in [7]) Let $\theta_{\mathcal{F}^{ij}}$ be the function introduced in [7, Appendix, Lemma A.1]. For all $\mathbf{u} \in \mathbf{W^{(i)}}$,*

$$|I^h(\theta_{\mathcal{F}^{ij}}\mathbf{u})|^2_{H^{1/2}(\partial\Omega_i)} \le C\left(1+\log\left(\frac{H_i}{h_i}\right)\right)^2\left(|\mathbf{u}|^2_{H^{1/2}(\partial\Omega_i)} + \frac{1}{H_i}\|\mathbf{u}\|^2_{L_2(\partial\Omega_i)}\right).$$

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