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Computational Mechanics
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Advanced Numerical Methods – Homework 10.

Exercise 1:
Let the ODE

$$y'(t) = -15\, y(t), \quad y(0) = 1.$$

be given.

1. Plot the exact solution for $t \in [0, 1]$.
2. Apply the explicit Euler scheme with step lengths $h = 1/4$ and $h = 1/8$ by hand or using MATLAB. Do the numerical solutions show the expected behavior? Relate this to the region of absolute stability of the explicit Euler method.
3. In the context of absolute stability, derive a maximal stable step length $h > 0$ for the given problem using the explicit Euler scheme.

Exercise 2:
Consider the decoupled system $y'(t) = Ay(t)$ with

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

with $\lambda_1, \lambda_2 \in \mathbb{C}$ and $\mathrm{Re}(\lambda_i) < 0, i = 1, 2$. Clearly, each component is a test equation.

1. Apply the explicit Euler scheme *independently* to each component. How would you choose the two different step sizes h_1, h_2 from the viewpoint of absolute stability? (You can draw this in a picture.)
2. If you want to apply the explicit Euler scheme to both equations *simultaneously*, you have to choose a common step size h.
   When does that matter and which one would you choose?
3. How does the picture change if you consider the implicit Euler method? You can argue geometrically.

Exercise 3:

a) Show that for a sufficiently many times differentiable function u it holds

$$\begin{aligned} u'' &= \tfrac{1}{h^2}[1 \quad -2 \quad 1]u + O(h^2) \\ &:= \tfrac{1}{h^2}[u(x-h) - 2u(x) + u(x+h)] + O(h^2). \end{aligned}$$

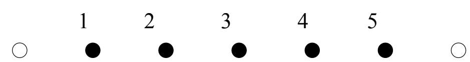
b) Using Taylor expansion, show directly that for a sufficiently many times differentiable function u it holds

$$\begin{aligned} \Delta u(x,y) &= \tfrac{\partial^2}{\partial x^2}u(x,y) + \tfrac{\partial^2}{\partial y^2}u(x,y) \\ &= h^{-2}[u(x-h,y) + u(x,y-h) - 4u(x,y) + u(x+h,y) + u(x,y+h)] + O(h^2) \end{aligned}$$

Programming Project 3 (4 Bonus Points)

1. Discretize the 1-D stationary heat equation

$$u'' = 1$$

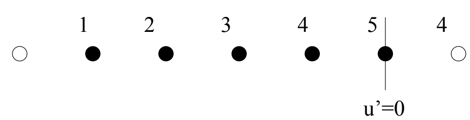


   with Dirichlet-boundary conditions $u = 0$ on the open circles. Use the finite difference approximation

$$u''(x) \approx \frac{1}{h^2}[u(x-h) - 2u(x) + u(x+h)].$$

   This leads to a system of linear equations.

2. Discretize the 1-D heat equation with Neumann boundary condition $u' = 0$ on the right side and Dirichlet boundary condition $u = 0$ on the left side.



3. Compute and plot both solutions using MATLAB.

Due date for Programming Project 3: 08/01/2013

Extended Hint: Implementation of the boundary conditions:
By $\partial\Omega_{\mathrm{D}} \subset \partial\Omega$ we denote the so-called *Dirichlet boundary* and by $\partial\Omega_{\mathrm{N}} \subset \partial\Omega$ the so-called *Neumann boundary* of the problem domain Ω.

Boundary conditions such as

$$u|_{\partial\Omega_{\mathrm{D}}} = 0, \quad \text{or} \quad u|_{\partial\Omega_{\mathrm{D}}} = -1$$

which prescribe the values of the solution on the boundary are called *Dirichlet boundary* conditions. These can be immediately inserted into the linear equation system arising from the finite difference approximation.

To model a fixed temperature on part of the boundary, one can also introduce isolating *Neumann boundary conditions*, i.e.,

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega_{\mathrm{N}}} = 0.$$

Note that in 1-D this reduces to $u'|_{\partial\Omega_{\mathrm{N}}} = \frac{\partial u}{\partial x}|_{\partial\Omega_{\mathrm{N}}} = 0$. Isolating boundary conditions can be imposed in a finite difference approximation by introducing so-called *ghost points*. One mirrors the outermost interior layer of discretization nodes next to the Neumann boundary **including their values** across the Neumann-boundary $\partial\Omega_{\mathrm{N}}$. This information can be inserted as additional equations into a linear equation system extended by the ghost points.

Note that this procedure is consistent with the finite difference approximation

$$u'(x) = \frac{u(x+h) - u(x-h)}{2h} + O(h^2)$$

on the Neumann boundary, i.e., for $x \in \partial\Omega_{\mathrm{N}}$.