Universität Duisburg-Essen
Computational Mechanics
Fakultät für Mathematik

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Summer Term 2013
05/30/2013

Advanced Numerical Methods – Homework 6.

Exercise 1
Consider the implicit scheme

$$y_{k+1} = y_k + \frac{h}{2}\left(f(x_k, y_k) + f(x_{k+1}, y_{k+1})\right).$$

Derive it from the trapezoidal quadrature rule and prove that its order of consistency is 2.

Exercise 2
Consider the following Butcher arrays:

$$\begin{array}{c|c} 0 & \\ \hline 1 & 1 \end{array} \qquad \begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline 1 & \frac{1}{2} & \frac{1}{2} \end{array} \qquad \begin{array}{c|ccc} 0 & & & \\ \frac{1}{2} & \frac{1}{2} & & \\ \frac{3}{4} & 0 & \frac{3}{4} & \\ \hline 1 & \frac{2}{9} & \frac{1}{3} & \frac{4}{9} \end{array} \qquad \begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline 1 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

Write down the corresponding Runge-Kutta schemes. If you recognize a scheme from the lecture write down the name.

Please turn the page.

Programming Project 2: (2 + 2 = 4 Bonus Points)

The following initial value problem describes the deflection of the central line $y(x)$ of a beam which is clamped at $x = 0$ and loaded by gravity (using idealized physical constants):

$$y''(x) = -(1 + y'(x)^2)^{3/2}, \qquad y(0) = 0, \quad y'(0) = 0,$$

for $x \in [0, 1]$.

1. Implement the Runge-Kutta scheme

$$\begin{aligned} k_1 &= f(x_i,\, y_i) \\ k_2 &= f(x_i + \frac{h}{2},\, y_i + \frac{h}{2}\, k_1) \\ k_3 &= f(x_i + \frac{h}{2},\, y_i + \frac{h}{2}\, k_2) \\ k_4 &= f(x_i + h,\, y_i + h k_3) \end{aligned}$$

$$\begin{aligned} x_{i+1} &= x_i + h \\ y_{i+1} &= y_i + h\,(\frac{1}{6}\, k_1 + \frac{1}{3}\, k_2 + \frac{1}{3}\, k_3 + \frac{1}{6}\, k_4) \end{aligned}$$

   to solve the second order IVP numerically.

2. Plot your numerical solution using the step lengths $h = 10^{-k}, k = 1, 2, 3, 4$ in one graph. Do the same for Heun's method.

3. Study the convergence of the two before-mentioned schemes using a loglog-plot of the global approximation error in $x = 0.8$ versus $1/h$. It is helpful to plot the functions $1/h, 1/h^2, 1/h^3, 1/h^4$ into the same figure! Try to read off the order of the convergence rate of the two schemes you implemented.

4. Derive the exact solution using the substitution $z = y'(x)$ and separation of variables.

How-To Turn In:

Please send your code to anm.uebung@uni-due.de in a .zip-archive. A worked out solution for parts 3. and 4. should be turned in on paper.

Hint: The exact solution for the IVP is given by

$$y(x) = \sqrt{1 - x^2} - 1, \quad x \in [0, 1].$$

Due Date: 06/13/2013 . 12:00h