Complex Numbers

The numbers everybody knows are the natural numbers and the integers.

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

But people learn in mathematic classes that it is not enough to have these numbers because they may solve equations like $4x = 20$. However when the solution of $4x = 3$ is required integers are no longer sufficient. Thus the fraction numbers are introduced, e.g., $\frac{3}{4}$ solves the equation. These numbers are then called rational numbers.
But as before the rational numbers are limited. The equations $x^2 = 4$ or $x^2 = \frac{9}{4}$ can be solved but $x^2 = 2$ cannot. Thus the irrational numbers as squareroot of 2 ($\sqrt{2}$) are introduced and together with the rational numbers we obtain the real numbers. For most computations the real numbers are enough to find solutions.
However if we are faced with an equation like $x^2 = -1$ we again can not find a solution within the real numbers, since all even powers of real numbers are positive. Thus the imaginary unit i is defined as the squareroot of (-1).
The astonishing thing is that after introducing the imaginary unit i all polynomials can be decomposed into linear factors, i.e., all polynomial equations can be solved. This fact is one of the main theorems of (linear) algebra.

Here, the complex numbers will be introduced as a twodimensional real vector space with a element-with-element multiplication, which gives a vector in the twodimensional real vector space $\mathbb{R}^2$ and with which $\mathbb{R}^2$ becomes a field.

Definition 1
A field F is a set on which an addition '$+$' and a multiplication '$\cdot$' are defined such that the following properties hold

C) *F is a* **closed** *set with respect to the defined addition and multiplication, i.e.,*

$$\forall a, b \in F \ : \ a + b \in F \ \text{ and } \ a \cdot b \in F.$$

Id) *There exist* **identity elements** *e_+ and $e_\cdot$ for the addition and the multiplication, i.e.,*

$$\forall a \in F \ : \ a + e_+ = a = e_+ + a \ \text{ and } \ a \cdot e_\cdot = a = e_\cdot \cdot a.$$

In) *There exist* **inverse elements** *with respect to the addition and the multiplication for all elements of F, i.e.,*

$$\forall a \in F \ \exists b = -a \ : \ a + b = e_+ \quad \text{and} \quad \forall a \in F \ \exists b = 1/a = a^{-1} \ : \ a \cdot b = e_\cdot$$

These operations indirectly define the subtraction and the division in the field.

A) *Addition and multiplication in F are* **associativ***, i.e.,*

$$\forall a, b, c \in F \ : \ a + (b + c) = (a + b) + c \ \text{ and } \ a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$

Co) *Addition and multiplication in F are* **commutative**, *i.e.,*

$$\forall a, b \in F \ : \ \ a + b = b + a \ \ \textit{and} \ \ a \cdot b = b \cdot a.$$

D) *The multiplication is* **distributive** *over the addition, i.e.,*

$$\forall a, b, c \in F \ : \ \ a \cdot (b + c) = (a \cdot b) + (a \cdot c).$$

The twodimensional real vector space is defined as

$$\mathbb{R}^2 = \left\{ (x, y)^T : x, y \in \mathbb{R} \right\} .$$

$\mathbb{R}^2$ becomes a field with the component-by-component defined addition, i.e.,

$$\forall \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in \mathbb{R}^2 \ : \ \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} := \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$$

and the multiplication defined by Definition 2.

Definition 2

$$\forall \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in \mathbb{R}^2 \ : \ \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} := \begin{pmatrix} x_1 x_2 - y_1 y_2 \\ x_1 y_2 + x_2 y_1 \end{pmatrix}$$

Lemma 1
For the mapping defined in Definition 2 the following properties hold

1. $\forall z_1 := (x_1, y_1)^T, z_2 := (x_2, y_2)^T \ : \ z_1 \cdot z_2 = z_2 \cdot z_1$ *(commutativity)*
2. $\forall z_1, z_2, z_3 := (x_3, y_3)^T \ : \ z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$ *(associativity)*
3. $\exists \ e. := (1, 0) \forall z := (x, y)^T \ : \ e. \cdot z = z \cdot e. = z$ *(identity element)*
4. $\forall z = (x, y) \neq (0, 0) \ \exists \ z^{-1} := \left(\frac{x}{x^2 + y^2}, -\frac{y}{x^2 + y^2} \right)$ *such that* $z \cdot z^{-1} = z^{-1} \cdot z = e.$ *(inverse element)*
5. $\forall z_1, z_2, z_3 \ : \ z_1 \cdot (z_2 + z_3) = (z_1 \cdot z_2) + (z_1 \cdot z_3)$ *(distributivity)*

Proof:

ad 1.

$$\begin{aligned} z_1 \cdot z_2 &= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 x_2 - y_1 y_2 \\ x_1 y_2 + x_2 y_1 \end{pmatrix} \\ &= \begin{pmatrix} x_2 x_1 - y_2 y_1 \\ x_2 y_1 + x_1 y_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = z_2 \cdot z_1 \end{aligned}$$

ad 2.

$$\begin{aligned} z_1 \cdot (z_2 \cdot z_3) &= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \cdot \begin{pmatrix} x_2x_3 - y_2y_3 \\ x_2y_3 + x_3y_2 \end{pmatrix} \\ &= \begin{pmatrix} x_1(x_2x_3 - y_2y_3) - y_1(x_2y_3 + x_3y_2) \\ x_1(x_2y_3 + x_3y_2) + (x_2x_3 - y_2y_3)y_1 \end{pmatrix} \\ &= \begin{pmatrix} (x_1x_2 - y_1y_2)x_3 - (x_1y_2 + x_2y_1)y_3 \\ (x_1x_2 - y_1y_2)y_3 + x_3(x_1y_2 + x_2y_1) \end{pmatrix} \\ &= \begin{pmatrix} x_1x_2 - y_1y_2 \\ x_1y_2 + x_2y_1 \end{pmatrix} \cdot \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = (z_1 \cdot z_2) \cdot z_3 \end{aligned}$$

ad 3.

$$\begin{aligned} e. \cdot z = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 1 \cdot x \\ 1 \cdot y \end{pmatrix} \\ &= \begin{pmatrix} x \cdot 1 \\ y \cdot 1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = z \cdot e. \end{aligned}$$

ad 4. For the corresponding inverse element to z holds $z^{-1} \cdot z = e. = z \cdot z^{-1}$. Since the commutativity was already shown we can determine $z^{-1} := (\tilde{x}, \tilde{y})$ using the first equality from above.

$$\begin{aligned} z^{-1} \cdot z &= e. \\ \Leftrightarrow \quad \begin{pmatrix} \tilde{x}x - \tilde{y}y \\ \tilde{x}y + x\tilde{y} \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \Leftrightarrow \quad \begin{cases} \tilde{x}x - \tilde{y}y = 1 \\ \tilde{x}y + x\tilde{y} = 0 \end{cases} & \\ \Rightarrow \quad \tilde{y} &= -\frac{\tilde{x}y}{x} \\ \Rightarrow \quad \tilde{x}x + \frac{\tilde{x}y}{x}y &= 1 \\ \Leftrightarrow \quad \tilde{x}(x^2 + y^2) &= x \\ \Leftrightarrow \quad \tilde{x} &= \frac{x}{x^2 + y^2} \\ \Rightarrow \quad \tilde{y} &= -\frac{y}{x^2 + y^2} \end{aligned}$$

ad 5.

$$\begin{aligned} z_1 \cdot (z_2 + z_3) &= \begin{pmatrix} x_1(x_2 + x_3) - y_1(y_2 + y_3) \\ x_1(y_2 + y_3) + (x_2 + x_3)y_1 \end{pmatrix} \\ &= \begin{pmatrix} x_1x_2 - y_1y_2 \\ x_1y_2 + x_2y_1 \end{pmatrix} + \begin{pmatrix} x_1x_3 - y_1y_3 \\ x_1y_3 + x_3y_1 \end{pmatrix} = (z_1 \cdot z_2) + (z_1 \cdot z_3) \qquad \square \end{aligned}$$

Theorem 1
The real vector space $\mathbb{R}^2$ equipped with the component-by-component addition and the multiplication defined in Definition 2 is a field. This field is called the <u>*field of complex numbers:*</u> *$\mathbb{C}$.*

The proof is left to the reader since it is very easy.

On $\mathbb{R}^2$ the multiplication with a scalar is defined

$$\forall \alpha \in \mathbb{R}, \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \ : \ \alpha \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
$$= \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \alpha e. \begin{pmatrix} x \\ y \end{pmatrix}$$

It can be shown with algebraic tools that the only n dimensional vector space which is a field is $\mathbb{C}$, i.e., it can be shown that for $n \geq 3$ no multiplication exists to obtain a field.

Euler introduced the imaginary unit as $i := (0,1)^T$. This leads to

$$i^2 = (0,1)^T \cdot (0,1)^T = (-1,0)^T$$

Furthermore $e = 1$ is written instead of $e := (1,0)^T$.
Thus every complex number can be written in the following form

$$\forall z = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C} \ : \ z = x + iy$$

For the multiplication in $\mathbb{C}$ one can now proceed as in the real numbers $\mathbb{R}$. Bearing in mind that $i^2 = -1$, this gives

$$(x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + iy_1x_2 + ix_1y_2 + i^2y_1y_2 = (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2)$$

as defined in Definition 2.

Definition 3
For $z = (x,y)^T = x + iy \in \mathbb{C}$ with $x, y \in \mathbb{R}$, the real and imaginary part are defined by

- $\mathrm{Re}(z) := x$ *as the* <u>*real part*</u> *of z,*
- $\mathrm{Im}(z) := y$ *as the* <u>*imaginary part*</u> *of z.*

If $\mathrm{Re}(z) := x = 0$, *z is called* <u>*purely imaginary,*</u> *and*
if $\mathrm{Im}(z) := y = 0$, *z is called* <u>*real.*</u>

To define the length of a vector in $\mathbb{C}$ which is then defined as its absolute value the complex conjugation is needed.

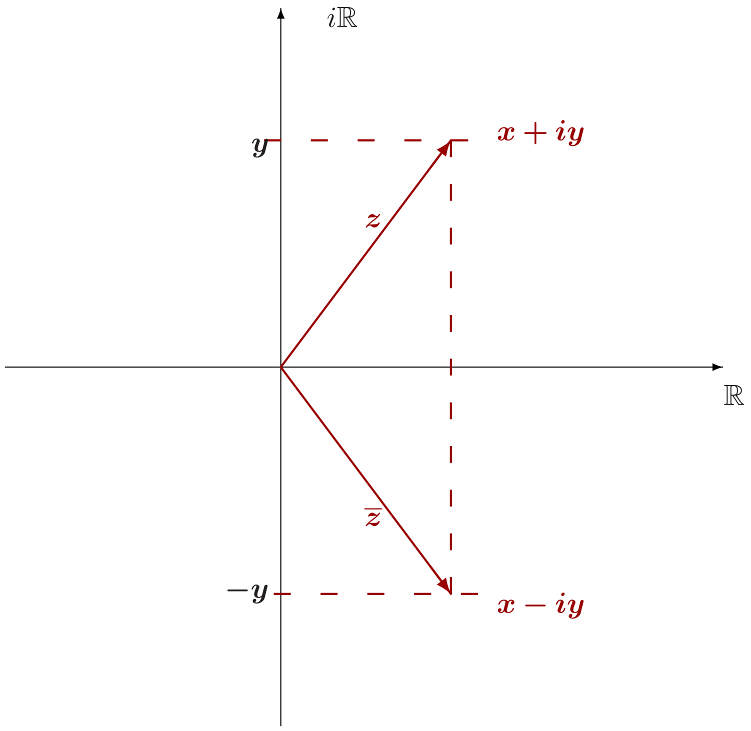

Figure 1: A complex number z and its complex conjugated number.

Definition 4
The mapping $\; : \mathbb{C} \to \mathbb{C}$ *defined by*

$$z = x + iy \mapsto \overline{z} := x - iy \;\; \forall x, y \in \mathbb{R}$$

defines the complex conjugation.

Remark 1
The complex conjugation defined in Definition 4 is an automorphism and an involution, i.e., it is an isomorphism from $\mathbb{C}$ *to* $\mathbb{C}$ *and the squared mapping gives the identity.*

For $z := x + iy$ we obtain

$$z\overline{z} = (x + iy)(x - iy) = x^2 - ixy + ixy - i^2y^2 = x^2 + y^2 \in \mathbb{R}_0^+,$$

with $\mathbb{R}_0^+ := \{x \in \mathbb{R} : x \geq 0\}$.

Definition 5
The absolute value of a complex number is defined as the Euclidean length of the twodimensional vector, i.e., the distance to origin.

$$\forall z := x + iy \in \mathbb{C} \quad : \quad |z| := \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}$$

Some properties of the absolute value of complex numbers are given in Theorem 2.

Theorem 2
The absolute value has the following properties

1. $\forall z \in \mathbb{C} \quad : \quad |z| = |\overline{z}|$ *and* $|z| = 0 \Leftrightarrow z = 0 = (0, 0)$

2. $\forall z_1, z_2 \in \mathbb{C} \ : \ |z_1 z_2| = |z_1|\ |z_2|$ *and if* $z_2 \neq 0$ *we also obtain* $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

3. $\forall z \in \mathbb{C} \ : \ -|z| \leq \mathrm{Re}(z) \leq |z|$ *and* $-|z| \leq \mathrm{Im}(z) \leq |z|$

4. $\forall z_1, z_2 \in \mathbb{C} \ : \ |z_1 + z_2| \leq |z_1| + |z_2|$ *(triangle inequality)*

Proof:

ad 1. $|z| = \sqrt{x^2+y^2} = \sqrt{x^2+(-y)^2} = |x+i(-y)| = |\overline{z}|$
$z\overline{z} = 0 \Leftrightarrow z = 0 \vee \overline{z} = 0 \Leftrightarrow x = 0 \wedge y = 0$

ad 2. $|z_1 z_2|^2 = (z_1 z_2)\overline{(z_1 z_2)} = z_1 z_2 \overline{z_1}\,\overline{z_2} = (z_1\overline{z_1})(z_2\overline{z_2}) = |z_1|\ |z_2|$

ad 3. The proof will be given only for the real part since the imaginary part can be treated analogously.
First assume that $x \geq 0$, then

$$\mathrm{Re}(z) = x = \sqrt{x^2} \leq \sqrt{x^2+y^2} = |z| \ \text{ since } \ x^2 \leq x^2 + y^2 \quad \forall y \in \mathbb{R}$$

and $-|z| \leq 0 \leq x$.
If $x \leq 0$ it follows that $x \leq 0 \leq |z|$ and

$$\mathrm{Re}(z) = x = -\sqrt{x^2} \geq -\sqrt{x^2+y^2} = -|z| \ \text{ since } \ x^2 \leq x^2 + y^2 \quad \forall y \in \mathbb{R}.$$

ad 4.

$$\begin{aligned}
|z_1+z_2|^2 = (z_1+z_2)\overline{(z_1+z_2)} &= (z_1+z_2)(\overline{z_1}+\overline{z_2}) \\
&= (z_1\overline{z_1}) + (z_1\overline{z_2}) + (z_2\overline{z_1}) + (z_2\overline{z_2}) \\
&= (z_1\overline{z_1}) + (z_1\overline{z_2}) + \overline{(\overline{z_2}z_1)} + (z_2\overline{z_2}) \\
&= |z_1|^2 + 2\mathrm{Re}(z_1\overline{z_2}) + |z_2|^2 \\
&\overset{3.}{\leq} |z_1|^2 + 2|z_1\overline{z_2}| + |z_2|^2 \\
&\overset{2.,1.}{\leq} |z_1|^2 + 2|z_1|\ |z_2| + |z_2|^2 \\
&= (|z_1|+|z_2|)^2 \qquad \square
\end{aligned}$$

There may be the question why the complex space is not introduced in the beginning. It is because on the real numbers there exists an orderrelation. But the complex numbers do not have such a relation.

One can also represent complex numbers in so-called polar coordinates.

Let $z := x + iy \neq 0 = 0 + i0$ with $x, y \in \mathbb{R}$, then

$$\exists!\ \varphi \in (-\pi, \pi]\ \exists!\ r \in (0, \infty) \ : \ z = r(\cos(\varphi) + i\sin(\varphi)).$$

Here $r = |z|$, $x = \text{Re}(z) = r\cos(\varphi)$ and $y = \text{Im}(z) = r\sin(\varphi)$, and the ! stands for *uniquely defined.*

Obviously φ may not always be uniquely defined on a larger interval since cosine and sine are alternating functions with period 2π.
Every φ which satisfies $z = r(\cos(\varphi) + i\sin(\varphi)), r > 0$ is called argument of z

$$\varphi = \arg(z).$$

If φ is limited to $(-\pi, \pi]$, φ is called the uniquely defined main value of the argument of $z \neq 0$.

This representation leads to a geometrical interpretation of the multiplication of two complex numbers. Furthermore it is helpfully when the division, n-th power and n-th root of a complex number should be defined.

Geometrical interpretation of Definition 2:

With $z_1 = |z_1|(\cos(\varphi_1) + i\sin(\varphi_1)) \neq 0$ and $z_2 = |z_2|(\cos(\varphi_2) + i\sin(\varphi_2)) \neq 0$ we have

$$\begin{aligned} z_1 z_2 &= |z_1|\,|z_2|\,\left[\cos(\varphi_1)\cos(\varphi_2) - \sin(\varphi_1)\sin(\varphi_2) + i\left(\cos(\varphi_1)\sin(\varphi_2) + \cos(\varphi_2)\sin(\varphi_1)\right)\right] \\ &= |z_1|\,|z_2|\,\left(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)\right). \end{aligned}$$

Here the angle sum identities for sine and cosine were used.

Example given approximately in Figure 2

$$\begin{aligned} z_1 = (3 + 2i) = \sqrt{13}(\cos(0.19\pi) + i\sin(0.19\pi)) \quad &\Rightarrow \quad \varphi_1 = 0.19\pi, |z_1| = \sqrt{13} \\ z_2 = (4 + i) = \sqrt{17}(\cos(0.08\pi) + i\sin(0.08\pi)) \quad &\Rightarrow \quad \varphi_2 = 0.08\pi, |z_2| = \sqrt{17} \end{aligned}$$

$$\begin{aligned} z_1 z_2 &= (3 + 2i)(4 + i) = 10 + 11i \\ &\Rightarrow \frac{10}{\sqrt{221}} = \cos(\phi) \Rightarrow \phi = 0.27\pi \\ z_1 z_2 &= \sqrt{13}(\cos(0.19\pi) + i\sin(0.19\pi))\sqrt{17}(\cos(0.08\pi) + i\sin(0.08\pi)) \\ &= \sqrt{221}(\cos(0.27\pi) + i\sin(0.27\pi)) \end{aligned}$$

Thus the multiplication of complex numbers can be interpreted as the multiplication of the absolute values (lengths) and the addition of the arguments.

Division of complex numbers:

For $z = |z|(\cos(\varphi) + i\sin(\varphi)) \neq 0$ one obtains

$$z^{-1} = \frac{1}{z} = \frac{1}{|z|(\cos(\varphi) + i\sin(\varphi))}$$

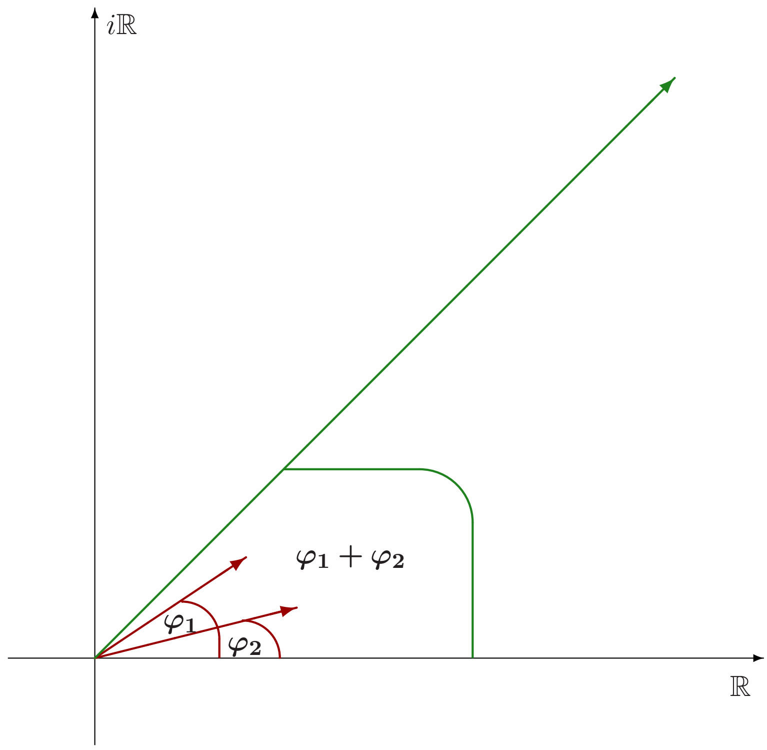

Figure 2: Geometrical interpretation of the multiplication of two complex numbers.

$$\begin{aligned}
&= \frac{1}{|z|} \underbrace{\frac{\cos(\varphi) - i\sin(\varphi)}{\cos^2(\varphi) + \sin^2(\varphi)}}_{=1} \\
&= \frac{\cos(\varphi) - i\sin(\varphi)}{|z|} \\
&= \frac{1}{|z|} \left(\cos(-\varphi) + i\sin(-\varphi)\right),
\end{aligned}$$

using an expansion with $\frac{\cos(\varphi)-i\sin(\varphi)}{\cos(\varphi)-i\sin(\varphi)}$ in the first step and using that cosine is symmetric and sine antisymmetric, i.e., $\cos(x) = \cos(-x)$ and $-\sin(x) = \sin(-x)$.

$$\Rightarrow \frac{z_1}{z_2} = z_1 z_2^{-1} = \frac{|z_1|}{|z_2|} \left(\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)\right)$$

Thus the division is the division of the absolute values and the subtraction of the arguments.

From the interpretation of the multiplication a formula for the n-th power can now easily obtained in polar coordinates.

n-th power of a complex number:

By induction the **de Moivre** formula is obtained

$$z^n = |z|^n \left(\cos(n\varphi) + i\sin(n\varphi)\right)$$

The de Moivre formula can be used to obtain the n-th root of a complex number.

n-th root of a complex number:

$w \in \mathbb{C}$ is called the n-th root of $z \quad :\Leftrightarrow \quad w^n = z$.
With $w = |w| \left(\cos(\psi) + i\sin(\psi)\right)$ this gives

$$\begin{aligned} |w|^n \left(\cos(n\psi) + i\sin(n\psi)\right) &= |z| \left(\cos(\varphi) + i\sin(\varphi)\right) \\ &= |z| \left(\cos(\varphi + 2\pi) + i\sin(\varphi + 2\pi)\right) \\ &\vdots \\ &= |z| \left(\cos(\varphi + 2(n-1)\pi) + i\sin(\varphi + 2(n-1)\pi)\right) \end{aligned}$$

Since for factors greater or equal n results equivalent to the ones for $i = 0, \ldots, n-1$ are achieved only the ones up to $n-1$ are considered.

Thus $|w|^n = |z|$ and $\quad n\psi = \varphi + 2k\pi \quad \Leftrightarrow \psi = \frac{\varphi}{n} + \frac{2k}{n}\pi \quad k = 0, \ldots, n$. From this it is clear that for $k = n$ the same results for $\cos(\psi)$ as for $k = 0$ and so on are obtained.

For every complex number $z = |z| \left(\cos(\varphi) + i\sin(\varphi)\right) \neq 0$ exist n n-th roots $w_0, \ldots, w_{n-1}$ with

$$w_k = \sqrt[n]{|z|} \left(\cos(\frac{\varphi + 2k\pi}{n}) + i\sin(\frac{\varphi + 2k\pi}{n}) \right) \quad k = 0, \ldots n-1$$

An example is given by, cf. Figure 3 for an approximate picture,

$$\begin{aligned} z = 4 + 4\sqrt{3}i \quad &\Rightarrow \quad |z| = 8\ ,\ \varphi = \frac{\pi}{3} = 60^\circ \\ &\Rightarrow \quad w_0^{(2)} = 2\sqrt{2}(\cos(\frac{\pi}{6}) + i\sin(\frac{\pi}{6})) = 2\sqrt{2}(\cos(30^\circ) + i\sin(30^\circ)) \\ & \quad w_1^{(2)} = 2\sqrt{2}(\cos(\frac{7\pi}{6}) + i\sin(\frac{7\pi}{6})) = 2\sqrt{2}(\cos(210^\circ) + i\sin(210^\circ)) \\ &\Rightarrow \quad w_0^{(3)} = 2(\cos(\frac{\pi}{9}) + i\sin(\frac{\pi}{9})) = 2(\cos(20^\circ) + i\sin(20^\circ)) \\ & \quad w_1^{(3)} = 2(\cos(\frac{7\pi}{9}) + i\sin(\frac{7\pi}{9})) = 2(\cos(140^\circ) + i\sin(140^\circ)) \\ & \quad w_2^{(3)} = 2(\cos(\frac{13\pi}{9}) + i\sin(\frac{13\pi}{9})) = 2(\cos(260^\circ) + i\sin(260^\circ)) \end{aligned}$$

The example shown in Figure 4 is for a real number. It shows approximately that there

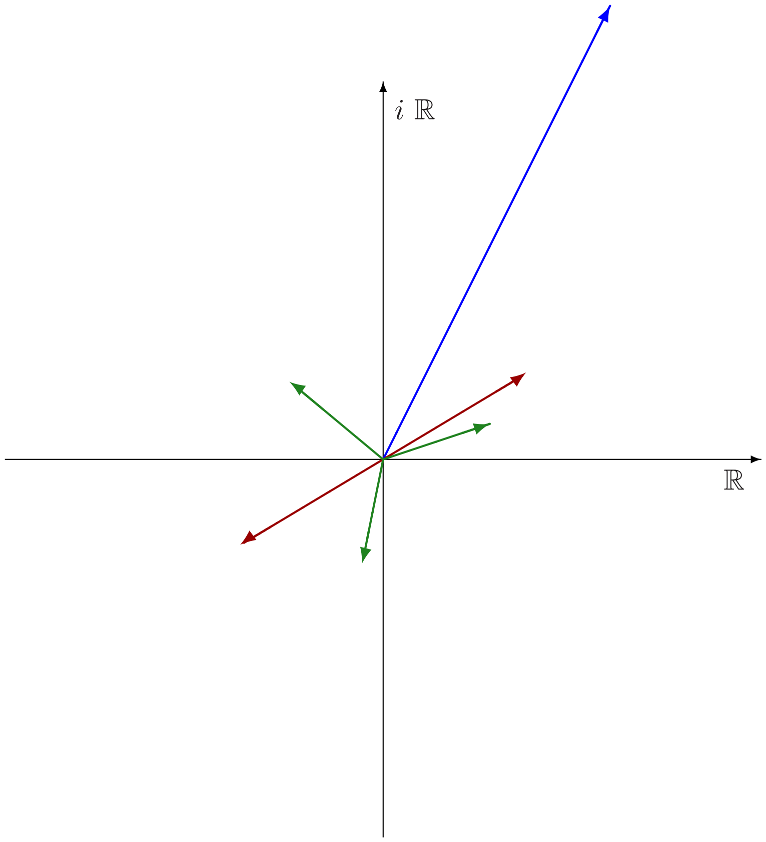

Figure 3: 2nd and 3rd root of a complex number a z.

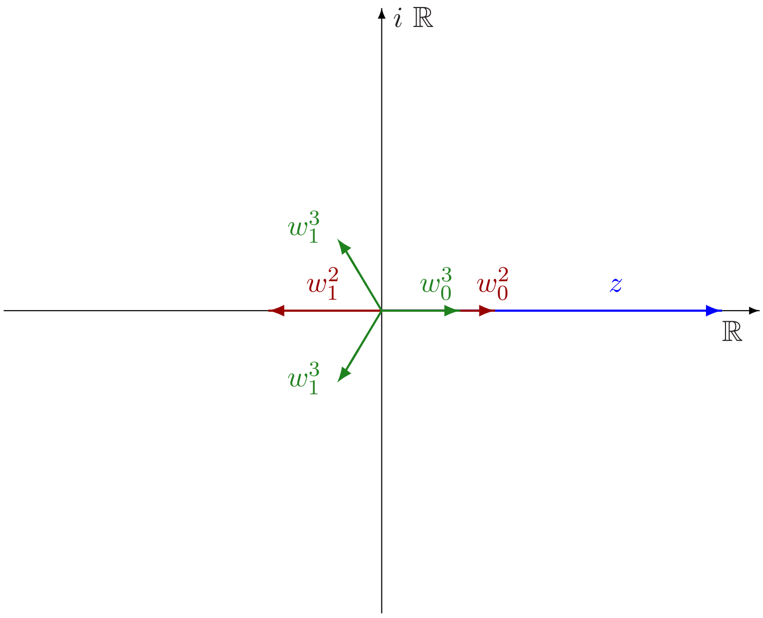

Figure 4: 2nd and 3rd root of a real number interpreted as a complex number z.

only exists one real third root but two more complex ones.

$$
\begin{aligned}
z = 9 \quad \Rightarrow \quad & |z| = 9 \ , \ \varphi = 0 = 0^\circ \\
\Rightarrow \quad & w_0^{(2)} = 3(\cos(0) + i\sin(0)) = 3 \\
& w_1^{(2)} = 3(\cos(\pi) + i\sin(\pi)) = -3 \\
\Rightarrow \quad & w_0^{(3)} = \sqrt[3]{9}(\cos(0) + i\sin(0)) = \sqrt[3]{9} \\
& w_1^{(3)} = \sqrt[3]{9}(\cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3})) = \sqrt[3]{9}(\cos(120^\circ) + i\sin(120^\circ)) \\
& w_2^{(3)} = \sqrt[3]{9}(\cos(\frac{4\pi}{3}) + i\sin(\frac{4\pi}{3})) = \sqrt[3]{9}(\cos(240^\circ) + i\sin(240^\circ))
\end{aligned}
$$