Lecture notes from the 23. December 2010

Please not that in the lecture it is section number 5 and not 0.
Bisection method

Let $f : [a, b] \to \mathbb{R}$ be continuous and with $f(a)f(b) < 0$.
Find $x^* \in [a, b]$ such that $f(x^*) = 0$.

$\overset{\text{intermediate value theorem}}{\Rightarrow}$ There exists at least one root of f in (a, b).

Algorithm 0.1 (Bisection)

$$
\begin{aligned}
x_p^{(0)} &:= \begin{cases} a & \text{if} \quad f(a) > 0 \\ b & \text{if} \quad f(b) > 0 \quad \text{(i.e., otherwise)} \end{cases} \\
\textit{and} \quad x_n^{(0)} &:= \begin{cases} a & \text{if} \quad f(a) < 0 \\ b & \text{if} \quad f(b) < 0 \quad \text{(i.e., otherwise)} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\text{for } k &= 0, \ldots \\
x_m^{(k)} &:= \frac{x_p^{(k)} + x_n^{(k)}}{2} \quad \text{(midpoint)} \\
x_p^{(k+1)} &:= \begin{cases} x_m^{(k)} & \text{if } f(x_m^{(k)}) > 0 \\ x_p^{(k)} & \text{otherwise} \end{cases} \\
x_n^{(k+1)} &:= \begin{cases} x_m^{(k)} & \text{if } f(x_m^{(k)}) < 0 \\ x_n^{(k)} & \text{otherwise} \end{cases}
\end{aligned}
$$

For $k \geq 0$ we have

1. $|x_p^{(k)} - x_n^{(k)}| = 2^{-k}(b - a) \quad \Rightarrow$ the length of the intervals converges to zero
   without loss of generality, we may assume $f(x_m^{(0)}) > 0$ which gives

$$
\begin{aligned}
|x_p^{(1)} - x_n^{(1)}| = |x_m^{(0)} - x_n^{(0)}| &= \left| \frac{x_p^{(0)} + x_n^{(0)}}{2} - x_n^{(0)} \right| \\
&= \left| \frac{x_p^{(0)} - x_n^{(0)}}{2} \right| \\
&= 2^{-1}(b - a)
\end{aligned}
$$

   Now we make an induction step $k \to k + 1$, assuming again without restriction $f(x_m^{(k)}) > 0$

$$
|x_p^{(k+1)} - x_n^{(k+1)}| = |x_m^{(k)} - x_n^{(k)}| = \left| \frac{x_p^{(k)} + x_n^{(k)}}{2} - x_n^{(k)} \right|
$$

$$
\begin{aligned}
&= \left| \frac{x_p^{(k)} - x_n^{(k)}}{2} \right| \\
&= 2^{-1} |x_p^{(k)} - x_n^{(k)}| \\
&= 2^{-1} 2^{-k} (b-a) \\
&= 2^{-(k+1)} (b-a)
\end{aligned}
$$

2. We assume without limitation $x_n^{(l)} < x_p^{(l)}$ for all l. Then

$$[x_n^{(k+1)}, x_p^{(k+1)}] \subset [x_n^{(k)}, x_p^{(k)}]$$

3. $f(x_p^{(k)}) \geq 0$, $f(x_n^{(k)}) \leq 0$ for all $k \geq 0$

Fixed point iteration
Remember: Find $x^* \in \mathbb{R}$ with $g(x^*) = x^*$

Algorithm 0.2 (Fixed point iteration)
Choose an initial guess $x^{(0)}$. For $k \geq 0$ compute

$$x^{(k+1)} := g(x^{(k)}).$$

Question: When does the fixed point iteration converge?

Newton iteration
Find x^* such that $f(x^*) = 0$.

Algorithm 0.3 (Newton iteration)
Choose an initial guess $x^{(0)}$. For $k \geq 0$ compute

$$x^{(k+1)} := x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}$$

The convergence of the Newton iteration depends on the initial value.

Secant method
Find x^* such that $f(x^*) = 0$.

Algorithm 0.4 (Secant method)
Let $x^{(0)}$ and $x^{(1)}$ be given. For $k \geq 1$ compute

$$x^{(k+1)} := x^{(k)} - f(x^{(k)}) \underbrace{\left(\frac{x^{(k)} - x^{(k-1)}}{f(x^{(k)}) - f(x^{(k-1)})} \right)}_{\text{inverse of the slope of the secant}}$$

0.1 Banach Fixed Point Theorem

Definition 0.1
A function $g : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is called **contractive** *in G with respect to the norm $\|\cdot\|$ if there exists a constant $q \in (0,1)$, i.e., $0 < q < 1$, such that*

$$\|g(x) - g(y)\| \leq q\|x - y\| \quad \forall x, y \in D. \tag{0.0}$$

The constant q is called **Lipschitz constant** *of g in D. A function g which satisfies (0.0), where the constant $q > 0$ may not be bounded from above by 1, is called* **Lipschitz continuous**.

Theorem 0.1
Let $D \subset \mathbb{R}^n$ be a convex set and the function $g : D \to \mathbb{R}^n$ differentiable. Furthermore, let $\boldsymbol{q := \sup_{x \in D} \|Dg(x)\| < 1}$*, where $Dg(x)$ is the Jacobi matrix*

$$Dg(x) = \begin{pmatrix} \frac{\partial g_1(x)}{\partial x_1} & \cdots & \frac{\partial g_1(x)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_n(x)}{\partial x_1} & \cdots & \frac{\partial g_n(x)}{\partial x_n} \end{pmatrix} \quad \text{of } g(x) = \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix}.$$

Then g is a contractive function.

Proof: <u>Remember:</u> $D \subset \mathbb{R}^n$ is convex means

$$\forall a, b \in D \Rightarrow x = ta + (1-t)b \in D \;\; \forall t \in [0,1]$$

Assume: $x, y \in D,\; t \in [0,1]$
Define: $f(t) := g(tx + (1-t)y)$
Chain rule: $\Rightarrow\; Df(t) = Dg(tx + (1-t)y)(x - y)$

$$\begin{aligned} \|g(x) - g(y)\| = \|f(1) - f(0)\| &= \|\int_0^1 Df(t)dt\| \\ &\leq \sup_{t\in[0,1]} \|Df(t)\| \\ &= \sup_{t\in[0,1]} \|Dg(tx + (1-t)y)(x-y)\| \\ &\leq \left(\sup_{t\in[0,1]} \|Dg(tx + (1-t)y)\|\right) \|(x-y)\| \\ &= \underbrace{\left(\sup_{z\in D} \|Dg(z)\|\right)}_{<1} \|(x-y)\| \end{aligned}$$

$\Rightarrow g$ is contractive. □

Definition 0.2 (Cauchy sequence)
A sequence $(a_n)_{n\in\mathbb{N}}$ *is called a* **Cauchy sequence** *or* **Cauchy convergent** *if and only if (:⇔)*

$$\begin{aligned} \forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \in \mathbb{N}, n \geq N \ \forall k \in \mathbb{N} \quad &: \quad \|a_n - a_{n+k}\| < \varepsilon \\ \forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n, m \in \mathbb{N}, n, m \geq N \quad &: \quad \|a_n - a_m\| < \varepsilon \end{aligned}$$

Theorem 0.2
For a sequence $(a_n)_{n\in\mathbb{N}}$ *in a complete space as* $\mathbb{R}$ *the following two statements are equivalent*

1. $(a_n)_{n\in\mathbb{N}}$ *is convergent.*
2. $(a_n)_{n\in\mathbb{N}}$ *is a Cauchy sequence.*