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OPTIMAL CONTROL OF THE THERMISTOR PROBLEM IN THREE SPATIAL DIMENSIONS, PART 1: EXISTENCE OF OPTIMAL SOLUTIONS*

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Abstract. This paper is concerned with the state-constrained optimal control of the three-dimensional thermistor problem, a fully quasilinear coupled system of a parabolic and elliptic PDE with mixed boundary conditions. This system models the heating of a conducting material by means of direct current. Local existence, uniqueness and continuity for the state system are derived by employing maximal parabolic regularity in the fundamental theorem of Prüss. Global solutions and controls admitting such are addressed and existence of optimal controls is shown if the temperature gradient is under control. This work is the first of two papers on the three-dimensional thermistor problem.

Key words. Partial differential equations, optimal control problems, state constraints

AMS subject classifications. 35K59, 35M10, 49J20

1. Introduction. In this paper, we consider the state-constrained optimal control of the three-dimensional thermistor problem. In detail the optimal control problem under consideration looks as follows:

$$\left.\begin{aligned} &\min \quad \frac{1}{2}\|\theta(T_1)-\theta_d\|^2_{L^2(E)}+\frac{\gamma}{s}\|\nabla\theta\|^s_{L^s(T_0,T_1;L^q(\Omega))}+\frac{\beta}{2}\int_{\Sigma_N}(\partial_t u)^2+|u|^p\,\mathrm{d}\omega\,\mathrm{d}t\\ &\text{s.t.} \quad (1.1)\text{–}(1.6)\\ &\text{and} \qquad \theta(x,t)\le\theta_{\max}(x,t) \quad \text{a.e. in } \Omega\times(T_0,T_1),\\ &\qquad\quad 0\le u(x,t)\le u_{\max}(x,t) \quad \text{a.e. on } \Gamma_N\times(T_0,T_1) \end{aligned}\right\} \quad \text{(P)}$$

where (1.1)–(1.6) refer to the following coupled PDE system consisting of the instationary nonlinear heat equation and the quasi-static potential equation, which is also known as *thermistor problem*:

$$\partial_t\theta-\operatorname{div}(\eta(\theta)\kappa\nabla\theta)=(\sigma(\theta)\rho\nabla\varphi)\cdot\nabla\varphi \qquad \text{in } Q:=\Omega\times(T_0,T_1) \tag{1.1}$$

$$\nu\cdot\eta(\theta)\kappa\nabla\theta+\alpha\theta=\alpha\theta_l \qquad \text{on } \Sigma:=\partial\Omega\times(T_0,T_1) \tag{1.2}$$

$$\theta(T_0)=\theta_0 \qquad \text{in } \Omega \tag{1.3}$$

$$-\operatorname{div}(\sigma(\theta)\rho\nabla\varphi)=0 \qquad \text{in } Q \tag{1.4}$$

$$\nu\cdot\sigma(\theta)\rho\nabla\varphi=u \qquad \text{on } \Sigma_N:=\Gamma_N\times(T_0,T_1) \tag{1.5}$$

$$\varphi=0 \qquad \text{on } \Sigma_D:=\Gamma_D\times(T_0,T_1). \tag{1.6}$$

Here θ is the temperature in a conducting material covered by the three dimensional domain Ω, while φ refers to the electric potential. The boundary of Ω is denoted by

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$\partial\Omega$ with the unit normal ν facing outward of Ω in almost every boundary point (w.r.t. the boundary measure ω). In addition, for the boundary we have $\Gamma_D \dot\cup \Gamma_N = \partial\Omega$, where Γ_D is closed within $\partial\Omega$. The functions $\eta(\cdot)\kappa$ and $\sigma(\cdot)\rho$ represent heat- and electric conductivity. While κ and ρ are given, prescribed matrix functions, η and σ are allowed to depend on the temperature θ. Moreover, α is the heat transfer coefficient regulating the heat flux through the boundary $\partial\Omega$, and θ_l and θ_0 are given boundary– and initial data, respectively. The quadratic gradient term in (1.1) is known as the Joule heat. Note that a realistic model of heat evolution includes a volumetric heat capacity $\varrho C_p(\theta)$, generally depending on θ, in front of the time derivative. We assume this term to be normalized to one, which can be achieved by re-scaling θ by so-called enthalpy transformation. The effects of this transformation on the remaining quantities in the equation may be absorbed into η, σ and α which does not influence the theory if C_p is reasonably smooth and strictly monotone (see e.g. [8, Sect. 3]). Finally, u stands for a current which is induced via the boundary part Γ_N and is to be controlled. The bounds in the optimization problem (P) as well as the desired temperature θ_d are given functions and β is the usual Tikhonov regularization parameter. The precise assumptions on the data in (P) and (1.1)–(1.6) will be specified in §2. In all what follows, the system (1.1)–(1.6) is frequently also called *state system*.

The PDE system (1.1)–(1.6) models the heating of a conducting material by means of an electric current, described by u, induced on the part Γ_N of the boundary, which is done for some time $T_1 - T_0$. At the grounding Γ_D, homogeneous Dirichlet boundary conditions are given, i.e., the potential is zero, inducing electron flow. Note that, usually, u will be zero on a subset Γ_{N_0} of Γ_N, which corresponds to having insulation at this part of the boundary. We emphasize that the different boundary conditions are essential for a realistic modeling of the process. The objective of (P) is to adjust the induced current u to minimize the L^2-distance between the desired and the resulting temperature at end time T_1 on the set $E \subseteq \Omega$, the latter representing the area of the material in which one is interested – realized in the objective functional by the first term. The other terms are present to minimize thermal stresses (second term) and to ensure a certain smoothness of the controls (third term), whose influence to the objective functional, however, may be controlled by the weights γ and β. The actual form of these terms and the size of the integrability orders are motivated by functional-analytic considerations, see §4. Moreover, the optimization is subject to pointwise control and state constraints. The control constraints reflect a maximum heating power, while the state constraints limit the temperature evolution to prevent possible damage, e.g. by melting of the material. Similarly to the mixed boundary conditions, the inequality constraints in (P) are essential for a realistic model as demonstrated by the numerical example in a companion paper [47]. Problem (P) is relevant in various applications, such as for instance the heat treatment of steel by means of an electric current. The numerical example mentioned deals with an application of this type.

1.1. Overview and main results. The optimal control problem (P) exhibits some non-standard features and challenges, in particular the quasilinear structure in both PDEs, including the nonlinear coupling and the nonlinear inhomogeneity in the heat equation (1.1). In contrast to the two-dimensional case (see [34]), it seems impossible to infer *a priori* bounds in suitable function spaces from the PDE system in three spatial dimensions. This absence of *a priori* bounds makes already the proof of existence and uniqueness of solutions to the state system challenging and rather technical; in particular, one cannot even guarantee existence of solutions which live globally in time in suitable regularity classes from properties of the state system only. Of course, the latter is also a problem for the optimal control setting because it conflicts with the point evaluation at T_1 in the objective functional; but even if one *had* global-in-time solutions, suitable *a priori* bounds are still mandatory for the proof of existence of optimal solutions to (P). For the optimal control problem however, we are able to circumvent this problem by deriving suitable *a priori* bounds using the objective functional and the state constraints, i.e., from additional information in the optimal control process. The functional-analytic framework is further complicated by the generally nonsmooth setting for the domain Ω and the coefficient matrices κ and ρ, and the regularity of φ is substantially limited by the presence of mixed boundary conditions for the elliptic equation in (1.5)–(1.6)

We next describe our main results and illustrate in some more detail how we deal with the above peculiarities. Our main result concerning existence and uniqueness of the state system (1.1)–(1.6) is established in §3 and relies on a fundamental theorem for abstract quasilinear evolution equations of Prüss [50] based on maximal parabolic regularity, cf. Proposition 3.17 below. The spatial function spaces will be negative Sobolev spaces which are readily adapted to account for mixed boundary conditions (see §2 for the relevant definitions). One of the central points to validate for Prüss' theorem—a fundamental issue when dealing with quasilinear equations—is to establish uniformity of the domains of the elliptic differential operators considered in these spaces; see Lemma 3.7. It turns out that exactly the regularity of θ needed to obtain uniform domains of the differential operators in the correct spaces also allows to show Lipschitz-continuity of the nonlinear inhomogeneities in (1.1) w.r.t. θ; this is done in the majority of §3.2, culminating in Propositions 3.21 and 3.28. The theorem of Prüss then yields our first main result (local-in-time solutions in a maximal regularity space, Theorem 3.14) which reads, informally, as follows:

Theorem. *For each $u \in L^{2r}(J; W^{-1,q}_{\Gamma_D}(\Omega))$ and $\theta_0 \in (W^{1,q}(\Omega), W^{-1,q}_{\emptyset}(\Omega))_{\frac{1}{r},r}$, there exists a unique local-in-time solution (θ, φ) of (1.1)–(1.6) with the regularity $\theta \in W^{1,r}(T_0, T_\bullet; W^{-1,q}_{\emptyset}(\Omega)) \cap L^r(T_0, T_\bullet; W^{1,q}(\Omega))$ and $\varphi \in L^{2r}(T_0, T_\bullet; W^{1,q}_{\Gamma_D}(\Omega))$ for some $T_\bullet \in J$, where q satisfies $q > 3$, and r is chosen large enough, depending on q.*

We refer to Assumption 3.4 and Definitions 3.11 and 3.12 for the precise requirements on q and r and the notion of a solution. Having the functional-analytic tools from §3 at hand, we next establish the existence of optimal controls to (P) in §4. Our local-in-time existence result and the blow-up examples in [6] indicate that one can in general not expect to obtain for every control u a solution of the state system (1.1)–(1.6) on the whole time interval. Therefore, the control-to-state mapping is in general not well defined as a mapping to a function space of solutions living on a fixed time interval, so that the standard proof of existence of an optimal control built upon a reduced formulation is not applicable in case of (P). To cope with this challenge, we restrict the set of feasible controls to the set $\mathcal{U}_g$ of controls admitting a *global-in-time* solution. This is justified because we are able to show that $\mathcal{U}_g$ is nonempty since it contains at least the zero control $u \equiv 0$, see Corollary 4.4. Nevertheless, attempting to perform the usual calculus of variations-proof for the existence of an optimal control to (P), we need to ensure that the infimal sequence of feasible points $(\theta_k, \varphi_k, u_k)$ admits a subsequence which converges in sufficiently strong spaces to be able to pass to the limit in the state system (1.1)–(1.6) which includes that the limit of this subsequence still gives rise to a global-in-time solution. As announced above, we use the objective functional and the state constraints in (P) to obtain the necessary bounds for the infimal sequence from the objective functional and the state constraints, because the state system itself admits no such bounds. We underline that, although the local-in-time existence result for the state system alone is not directly used to establish the existence of a solution to the optimal control problem (P), most of the results in §3 also play an essential role for the derivation of suitable bounds for the infimal sequence. It is then shown in Theorem 4.7 and Lemma 4.12 how these bounds translate to convergence of a suitable subsequence with the desired properties. These results, together with continuity properties for the state equation established in [48], are finally used for our second main theorem, the existence of optimal "global" controls for (P), in Theorem 4.14:

THEOREM. *There exists an optimal solution* $(\bar\theta, \bar\varphi, \bar u)$ *to* (P).

Note that this non-standard technique to establish *a priori* bounds for the infimal sequence was also used to prove existence of optimal controls in [4]. In a companion paper [47] we moreover show that $\mathcal{U}_g$ is in fact an open set, which together with the continuity of θ allows to derive necessary optimality conditions in qualified form for (P) in which $\mathcal{U}_g$ does not explicitly occur.

1.2. Context and related works. Let us put our work into perspective. Up to the authors' best knowledge, there are only few contributions dealing with the optimal control of the thermistor problem. We refer to [44, 14, 37, 36], where two-dimensional problems are discussed. In [44], a completely parabolic problem is discussed, while [37] considers the purely elliptic counterpart to (1.1)–(1.6). In [14, 5, 36], the authors investigate a parabolic-elliptic system similar to (1.1)–(1.6), assuming a particular structure of the controls. In contrast to [44, 37], mixed boundary conditions are considered in [14, 36]. However, all these contributions do not consider pointwise state constraints and non-smooth data. Thus, (P) differs significantly from the problems considered in the aforementioned papers. In a previous paper [34], two of the authors investigated the two-dimensional counterpart of (P). This contribution also accounts for mixed boundary conditions, non-smooth data, and pointwise state constraints. However, the analysis in [34] substantially differs from the three dimensional case considered here. First of all, in two spatial dimensions, the isomorphism-property of the elliptic operators mentioned above directly follows from the classical paper [26]. Moreover, the heat conduction coefficient in (1.1) is assumed not to depend on the temperature in [34]. Both features allow to derive suitable *a priori* bounds from the PDE system itself, i.e., the essential feature that is missing in the three dimensional setting considered here, as already explained above. This enables to establish a global existence result for a suitable class of control functions without deriving necessary *a priori* bounds from a particularly chosen objective as in our case. Hence, main aspects of the present work do not appear in the two-dimensional setting. Let us finally take a broader look on state-constrained optimal control problems governed by PDEs. Compared to semilinear state-constrained optimal control problems, the literature concerning optimal control problems subject to quasilinear PDEs and pointwise state constraints is rather scarce. We exemplarily refer to [12, 11], where elliptic problems are studied. The vast majority of papers in this field deals with problems that possess a well defined control-to-state operator. By contrast, as indicated above, the state-system (1.1)–(1.6) in general just admits local-in-time solutions, which requires a sophisticated treatment of the optimal control problem under consideration.

1.3. Outline of the paper. The paper is organized as follows: We set the stage with notations and assumptions in §2 and discuss the state-system in §3. More precisely, §3.1 collects preliminary results, also interesting for their own sake, while §3.2

is devoted to the actual proof of existence and uniqueness of local-in-time solutions. We then proceed with the optimal control problem in §4, give sufficient conditions for sets of controls to be closed within the sets of all controls which admit global solutions in time, and finally show that optimal solutions to (P) exist.

2. Notations and general assumptions. We introduce some notation and the relevant function spaces. All function spaces under our consideration are *real* ones. Let, for now, Ω be a domain in $\mathbb{R}^3$. We give precise geometric specifications for Ω in §2.1 below.

Let us fix some notations: The underlying time interval is called $J = (T_0, T_1)$ with $T_0 < T_1$. The boundary measure for the domain Ω is called ω. Generally, given an integrability order $q \in (1, \infty)$, we denote the conjugate of q by q', i.e., it always holds $1/q + 1/q' = 1$.

Definition 2.1. *For $q \in (1, \infty)$, let $W^{1,q}(\Omega)$ denote the usual Sobolev space on Ω. If $\Xi \subset \partial\Omega$ is a closed part of the boundary $\partial\Omega$, we set $W^{1,q}_\Xi(\Omega)$ to be the closure of the set $\{\psi|_\Omega : \ \psi \in C_0^\infty(\mathbb{R}^3),\ \mathrm{supp}\ \psi \cap \Xi = \emptyset\}$ with respect to the $W^{1,q}$-norm.*

The dual space of $W^{1,q'}_\Xi(\Omega)$ is denoted by $W^{-1,q}_\Xi(\Omega)$; in particular, we write $W^{-1,q}_\emptyset(\Omega)$ for the dual of $W^{1,q'}(\Omega)$ (see Remark 2.2 below regarding consistency). The Hölder spaces of order δ on Ω or order ϱ on Q are denoted by $C^\delta(\Omega)$ and $C^\varrho(Q)$, respectively (note here that Hölder continuous functions on Ω or Q, respectively, possess an unique uniformly continuous extension to the closure of the domain, such that we will mostly use $C^\delta(\overline{\Omega})$ and $C^\varrho(\overline{Q})$ to emphasize on this).

We will usually abbreviate the function spaces on Ω by leaving out the Ω, e.g. we write $W^{1,q}_\Xi$ instead of $W^{1,q}_\Xi(\Omega)$ or L^p instead of $L^p(\Omega)$. Lebesgue spaces on subsets of $\partial\Omega$ are always to be considered with respect to the boundary measure ω, but we abbreviate $L^p(\partial\Omega, \omega)$ by $L^p(\partial\Omega)$ and do so analogously for any ω-measurable subset of the boundary. The norm in a Banach space X will be always indicated by $\|\cdot\|_X$. For two Banach spaces X and Y, we denote the space of linear, bounded operators from X into Y by $\mathcal{L}(X;Y)$. The symbol $\mathcal{LH}(X;Y)$ stands for the set of linear homeomorphisms between X and Y. If X, Y are Banach spaces which form an interpolation couple, then we denote by $(X, Y)_{\tau,r}$ the real interpolation space, see [54]. We use $\mathbb{R}^{3\times 3}_{\mathrm{sym}}$ for the set of real, symmetric 3×3-matrices. In the sequel, a linear, continuous injection from X to Y is called an *embedding*, abbreviated by $X \hookrightarrow Y$. For Lipschitz continuous functions f, we denote the Lipschitz constants by L_f, while for bounded functions g we denote their bound by M_g (both over appropriate sets, if necessary). Finally, c denotes a generic positive constant.

2.1. Geometric setting for Ω and Γ_D. In all what follows, the symbol Ω stands for a bounded Lipschitz domain in $\mathbb{R}^3$ in the sense of [46, Ch. 1.1.9]; cf. [33] for the boundary measure ω on such a domain. The thus defined notion is different from *strong Lipschitz domain*, which is more restrictive and in fact identical with *uniform cone domain*, see again [46, Ch. 1.1.9]). A Lipschitz domain is formed e.g. by the topologically regularized union of two crossing beams (see [31, Ch. 7]), which is *not* a strong Lipschitz domain. Moreover, the interior of any three-dimensional connected polyhedron is a Lipschitz domain, if the polyhedron is, simultaneously, a 3-manifold with boundary, cf. [30, Thm. 3.10]. However, a ball minus half of the equatorial plate is *not* a Lipschitz domain, and a chisel, where the blade edge is bent onto the disc, is also not.

Remark 2.2 (Extension property). *The Lipschitz property of Ω implies the existence of a linear, continuous extension operator $\mathfrak{E}\colon W^{1,q}(\Omega) \to W^{1,q}(\mathbb{R}^3)$ (see [22,* *p.165]), which simultaneously provides a continuous extension operator $\mathfrak{E}\colon C^\delta(\Omega) \to C^\delta(\mathbb{R}^3)$ and $\mathfrak{E}\colon L^p(\Omega) \to L^p(\mathbb{R}^3)$, where $\delta \in (0,1)$ and $p \in [1,\infty]$. This has the following consequences:*

(i) *Since any element from $W^{1,q}(\mathbb{R}^3)$ may be approximated by smooth functions in the $W^{1,q}$-norm, any element from $W^{1,q}(\Omega)$ may be approximated by restrictions of smooth functions in the $W^{1,q}(\Omega)$-norm. This tells us that the definitions of $W^{1,q}(\Omega)$ and $W^{1,q}_\Xi(\Omega)$ are consistent in case of $\Xi = \emptyset$, i.e., one has $W^{1,q}(\Omega) = W^{1,q}_\emptyset(\Omega)$. See also the detailed discussion in [25, Ch. 1.3.2].*

(ii) *The existence of the extension operator $\mathfrak{E}$ provides the usual Sobolev embeddings, that is, $W^{1,q}(\Omega) \hookrightarrow L^p(\Omega)$ for $1 + 3/p \geq 3/q$, and their consequences. For our work, a particularly critical consequence is the embedding $L^{q/2}(\Omega) \hookrightarrow W^{-1,q}_\emptyset(\Omega)$ if q exceeds the space dimension three. Moreover, we also have the usual boundary embeddings or trace theorems at our disposal, see [35, Lemma 2.7].*

Next we define the geometric setting for the domain Ω and the Dirichlet boundary part. For this, we use the following model sets, based on the open unit cube $K = (-1,1)^3$ in $\mathbb{R}^3$, centered at $0 \in \mathbb{R}^3$:

$$
\begin{aligned}
K_-\colon &= \{\mathrm{x} \in K \colon x_3 < 0\} && \text{(lower half cube)},\\
\Sigma_K\colon &= \{\mathrm{x} \in K \colon x_3 = 0\} && \text{(upper plate of } K_-),\\
\Sigma_K^0\colon &= \{\mathrm{x} \in \Sigma_K \colon x_2 < 0\} && \text{(left half of } \Sigma_K).
\end{aligned}
$$

The definition is then as follows:

Definition 2.3 (Regular sets). *Let Ω be a bounded Lipschitz domain and let $\Xi \subset \partial\Omega$ be closed within $\partial\Omega$.*

(i) *We say that $\Omega \cup \Xi$ is* regular *(in the sense of Gröger), if for any point $\mathrm{x} \in \partial\Omega$ there is an open neighborhood U_x of x, a number $a_\mathrm{x} > 0$ and a bi-Lipschitz mapping ϕ_x from U_x onto $a_\mathrm{x}K$ such that $\phi_\mathrm{x}(\mathrm{x}) = 0 \in \mathbb{R}^3$, and we have either $\phi_\mathrm{x}\big((\Omega \cup \Xi) \cap U_\mathrm{x}\big) = a_\mathrm{x}K_-$ or $a_\mathrm{x}(K_- \cup \Sigma_K)$ or $a_\mathrm{x}(K_- \cup \Sigma_K^0)$.*

(ii) *The regular set $\Omega \cup \Xi$ is said to satisfy the* volume-conservation condition*, if each mapping ϕ_x in Condition (i) is volume-preserving.*

Generally, Ξ is allowed to be empty in Definition 2.3. Then Definition 2.3 (i) merely describes a Lipschitz domain. Some further comments are in order:

Remark 2.4 (Comments on regular sets).

(i) *Condition (i) in Definition 2.3 exactly characterizes Gröger's* regular sets*, introduced in his pioneering paper [26]. Note that the volume-conservation condition also has been required in several contexts, cf. [23] and [27].*
Clearly, the properties $\phi_\mathrm{x}(U_\mathrm{x}) = a_\mathrm{x}K$ and $\phi_\mathrm{x}\big(\Omega \cap U_\mathrm{x}\big) = a_\mathrm{x}K_-$ are already ensured by the Lipschitz property of Ω; the crucial point is the behavior of $\phi_\mathrm{x}(\Xi \cap U_\mathrm{x})$.

(ii) *A simplifying topological characterization of Gröger's regular sets in the case of three space dimensions reads as follows (cf. [32, Ch. 5]):*

1. *Ξ is the closure of its interior within $\partial\Omega$,*
2. *the boundary $\partial\Xi$ within $\partial\Omega$ is locally bi-Lipschitz diffeomorphic to the open unit interval $(0,1)$.*

(iii) *In particular, all domains with Lipschitz boundary (synonymous:* strong *Lipschitz domains) satisfy Definition 2.3: if, after a shift and an orthogonal transformation, the domain lies locally beyond a graph of a Lipschitz function ψ, then one can define $\phi(x_1,x_2,x_3) = (x_1 - \psi(x_2,x_3), x_2, x_3)$. Obviously, the mapping ϕ is then bi-Lipschitz and the determinant of its Jacobian is identically 1.*

(iv) *It turns out that regularity together with the volume-conservation condition is not a too restrictive assumption on the mapping ϕ_x. In particular, there are such* *mappings—although not easy to construct—which map the ball onto the cylinder, the ball onto the cube and the ball onto the half ball, see [24, 19]. The general message is that this class has enough flexibility to map "non-smooth" objects onto smooth ones.*

(v) *If Ξ is nonempty and $\Omega \cup \Xi$ is regular, then Ξ has interior points (with respect to the boundary topology in $\partial\Omega$), and, consequently, never has boundary measure 0.*

The following assumption is supposed to be valid for all the remaining considerations in the paper.

ASSUMPTION 2.5. *The set $\Omega \cup \Gamma_D$ is regular with $\Gamma_D \neq \emptyset$.*

For the moment, it is sufficient to impose only the regularity condition from Assumption 2.5 (i) on $\Omega \cup \Gamma_D$. The volume-conservation condition is not needed until §4, cf. Assumption 4.2 below. As explained in Remark 2.4, Assumption 2.5 in particular implies that $\omega(\Gamma_D) > 0$.

2.2. General assumptions on (P). We first address the assumptions regarding (local) existence and uniqueness for the state equation (1.1)–(1.6). This means in particular that we treat u as a fixed, given inhomogeneity in this context, whereas it is an unknown control function when considering the optimal control problem (P).

ASSUMPTION 2.6 (State system). *On the quantities in the state system (1.1)–(1.6) we generally impose:*

(i) *The functions $\sigma : \mathbb{R} \to (0,\infty)$ and $\eta \colon \mathbb{R} \to (0,\infty)$ are bounded and Lipschitzian on any bounded interval,*

(ii) *the function $\rho \in L^\infty(\Omega; \mathbb{R}^{3\times 3}_{sym})$ takes* symmetric *matrices as values, and satisfies the usual ellipticity condition, i.e.,*

$$\operatorname*{ess\,inf}_{x\in\Omega} \sum_{i,j=1}^{3} \rho_{ij}(x)_{ij}\, \xi_i\, \xi_j \geq \underline{\rho}\, \|\xi\|^2_{\mathbb{R}^3} \quad \forall\, \xi \in \mathbb{R}^3$$

with a constant $\underline{\rho} > 0$,

(iii) *the function $\kappa \in L^\infty(\Omega; \mathbb{R}^{3\times 3}_{sym})$ also takes* symmetric *matrices as values, and, additionally, satisfies an ellipticity condition, that is,*

$$\operatorname*{ess\,inf}_{x\in\Omega} \sum_{i,j=1}^{3} \kappa_{ij}(x)\, \xi_i\, \xi_j \geq \underline{\kappa}\, \|\xi\|^2_{\mathbb{R}^3} \quad \forall\, \xi \in \mathbb{R}^3$$

holds with a constant $\underline{\kappa} > 0$,

(iv) *$\theta_l \in L^\infty(J; L^\infty(\partial\Omega))$,*

(v) *$\alpha \in L^\infty(\partial\Omega)$ with $\alpha(x) \geq 0$ a.e. on $\partial\Omega$ and $\int_{\partial\Omega} \alpha\, d\omega > 0$,*

(vi) *$u \in L^{2r}(J; W^{-1,q}_{\Gamma_D})$ for some $q > 3$ to be specified in Assumption 3.4 below and $r > \frac{2q}{q-3}$, cf. Definition 3.11 and Theorem 3.14 below,*

(vii) *$\theta_0 \in (W^{1,q}, W^{-1,q}_{\emptyset})_{\frac{1}{r},r}$ with q and r as in (vi).*

REMARK 2.7. *In assumption (vi), we implicitly made use of the embedding $L^{\mathfrak{p}}(\Gamma_N) \hookrightarrow W^{-1,q}_{\Gamma_D}$ for $\mathfrak{p} > \frac{2}{3}q$, realized by the adjoint operator of the continuous trace operator $\tau_{\Gamma_N} \colon W^{1,q'}_{\Gamma_D} \to L^{\mathfrak{p}'}(\Gamma_N)$, cf. Remark 2.2. In this sense, a function $u \in L^{2r}(J; L^{\mathfrak{p}}(\Gamma_N))$ is considered as an element of $L^{2r}(J; W^{-1,q}_{\Gamma_D})$. In the same manner, we will treat the function $\alpha\theta_l \in L^\infty(J; L^\infty(\partial\Omega))$ as an element of $L^\infty(J; W^{-1,q}_{\emptyset})$.*

Next we turn to the assumptions concerning the optimal control problem (P). Now, u plays the role of the searched-for variable or function, whose regularity is implicitly determined by the objective functional in (P). As we will see in the sequel in

§4, our hypotheses on the objective functional stated below imply that the restriction of the optimal control problem to control functions from a function space $\mathbb{U}$ compatible with the control term in the objective functional yields the desired properties such as a continuous embedding into $L^{2r}(J; W^{-1,q}_{\Gamma_D})$ as required in Assumption 2.6 (vi), see (4.4) and Lemma 4.12 below.

ASSUMPTION 2.8 (Optimal control problem). *The remaining quantities in* (P) *fulfill:*

(i) *The integrability exponents in the objective functional satisfy* $p > \frac{4}{3}q - 2$ *and* $s > \frac{2q}{q-3}(1 - \frac{3}{q} + \frac{3}{\varsigma})$*, where* q *and* ς *are specified in Assumption 3.4 and Definition 4.6 below.*

(ii) E *is an open (not necessarily proper) subset of* Ω*.*

(iii) $\theta_d \in L^2(E)$*.*

(iv) $\theta_{\max} \in C(\overline{Q})$ *with* $\max(\max_{\overline{\Omega}} \theta_0, \operatorname{ess\,sup}_\Sigma \theta_l) \le \theta_{\max}(x,t)$ *for all* $(x,t) \in \overline{Q}$ *and* $\theta_0(x) < \theta_{\max}(x, T_0)$ *for all* $x \in \overline{\Omega}$*.*

(v) $u_{\max}$ *is a given function with* $u_{\max}(x,t) \ge 0$ *a.e. on* Σ_N*.*

(vi) $\beta > 0$.

Note that we do not impose any regularity assumptions on the function $u_{\max}$. In particular, it is allowed that $u_{\max} \equiv \infty$ so that no upper bound is present.

3. Rigorous formulation, existence and uniqueness of solutions for the thermistor problem.

In this chapter we will present a precise analytical formulation for the thermistor-problem, see Definition 3.12 below. In order to do so, we first recall some background material. One of the most crucial points is the requirement of suitable mapping property for Poisson's operator, cf. Assumption 3.4. The reader should note that a similar condition was also posed in [6, Ch. 3] in order to get smoothness of the solution; compare also [20], where exactly this regularity for the solution of Poisson's equation is needed in order to show uniqueness for the semiconductor equations. We prove, in particular, some preliminary results which are needed later on and which may be also of independent interest. After having properly defined a solution of the thermistor problem, we establish some more preparatory results and afterwards show existence (locally in time) and uniqueness of the solution of the thermistor problem in Section 3.2. Finally, we show that our concept to treat the problem is not accidental, but—more or less—inevitable.

3.1. Prerequisites: Elliptic and parabolic regularity.

We begin this subsection with the definition of the divergence operators. First of all, let us introduce the brackets $\langle \cdot, \cdot \rangle$ as the symbol for the dual pairing between $W^{-1,2}_\Xi$ and $W^{1,2}_\Xi$, extending the scalar product in L^2.

DEFINITION 3.1 (Divergence-gradient operator). *Let* $\Xi \subset \partial\Omega$ *be closed. Assume that* μ *is any bounded, measurable,* $\mathbb{R}^{3\times 3}_{sym}$*-valued function on* Ω *and that* $\gamma \in L^\infty(\partial\Omega \setminus \Xi)$ *is nonnegative. We define the operators* $-\nabla \cdot \mu \nabla$ *and* $-\nabla \cdot \mu\nabla + \tilde{\gamma}$*, each mapping* $W^{1,2}_\Xi$ *into* $W^{-1,2}_\Xi$*, by*

$$\big\langle -\nabla \cdot \mu \nabla \psi, \xi \big\rangle := \int_\Omega (\mu \nabla \psi) \cdot \nabla \xi \,\mathrm{d}x \quad \textit{for} \quad \psi, \xi \in W^{1,2}_\Xi$$

and

$$\big\langle (-\nabla \cdot \mu \nabla + \tilde{\gamma})\psi, \xi \big\rangle = \big\langle -\nabla \cdot \mu \nabla \psi, \xi \big\rangle + \int_{\partial\Omega\setminus\Xi} \gamma\, \psi\, \xi \,\mathrm{d}\omega \quad \textit{for} \quad \psi, \xi \in W^{1,2}_\Xi. \tag{3.1}$$

In all what follows, we maintain the same notation for the corresponding maximal restrictions to $W^{-1,q}_\Xi$*, where* $q > 2$*, and denote the domain for the operator* $-\nabla \cdot \mu \nabla$*,* *when restricted to* $W^{-1,q}_\Xi$, *by* $\mathcal{D}_q(\mu)$, *equipped with the graph norm.*

REMARK 3.2. *The estimate*

$$\big\|-\nabla\cdot\mu\nabla\psi\big\|_{W^{-1,q}_\Xi} = \sup_{\|\varphi\|_{W^{1,q'}_\Xi}=1} \left|\int_\Omega (\mu\nabla\psi)\cdot\nabla\varphi\,\mathrm{d}x\right| \le \|\mu\|_{L^\infty}\|\psi\|_{W^{1,q}_\Xi} \tag{3.2}$$

shows that $W^{1,q}_\Xi$ *is embedded in* $\mathcal{D}_q(\mu)$ *for every bounded coefficient function* μ. *It is also known that* $\mathcal{D}_q(\mu)\hookrightarrow C^\alpha(\overline{\Omega})$ *for some* $\alpha>0$ *whenever* $q>3$, *see [32, Thm. 3.3]. Additionally,* (3.2) *implies that the mapping*

$$L^\infty(\Omega;\mathbb{R}^{3\times 3}_{sym}) \ni \mu \mapsto \nabla\cdot\mu\nabla \in \mathcal{L}(W^{1,q}_\Xi;W^{-1,q}_\Xi)$$

is a linear and continuous contraction for every $q\in(1,\infty)$.

In the following, we consider the operators defined in Definition 3.1 mostly in two incarnations: firstly, the case $\Xi=\emptyset$ and $\mu=\kappa$; and secondly $\Xi=\Gamma_D$ with $\mu=\rho$. We write $-\nabla\cdot\kappa\nabla$ and $-\nabla\cdot\kappa\nabla+\tilde{\alpha}$ in the first, and $-\nabla\cdot\rho\nabla$ in the second case. The next result from [26, 28] is a fundamental maximal elliptic regularity assertion for $-\nabla\cdot\mu\nabla$ in the $W^{1,q}_\Xi$ setting for $q>2$. In particular, it asserts the crucial *a priori* bounds uniformly in the coefficient functions for these q, which was sufficient for the treatment in two space dimensions in [34]. We will make use of these uniform bounds in Proposition 4.7 below.

PROPOSITION 3.3 ([26, 28]). *Let* $\Omega\cup\Xi$ *be regular in the sense of Definition 2.3, let* μ,γ *be as in Definition 3.1 and suppose that either* $\omega(\Xi)>0$ *or* $\Xi=\emptyset$ *and* $\int_{\partial\Omega}\gamma\,\mathrm{d}\omega>0$. *Then there is a number* $q_0>2$ *such that*

$$-\nabla\cdot\mu\nabla+\tilde{\gamma}\colon W^{1,q}_\Xi\to W^{-1,q}_\Xi$$

is a topological isomorphism for all $q\in[2,q_0]$. *The number* q_0 *may be chosen uniformly for all coefficient functions* μ *with the same ellipticity constant and the same* L^∞*-bound. Moreover, for each* $q\in[2,q_0]$, *the norm of the inverse of* $\nabla\cdot\mu\nabla+\tilde{\gamma}$ *as a mapping from* $W^{-1,q}_\Xi$ *to* $W^{1,q}_\Xi$ *may be estimated again uniformly for all coefficient functions with the same ellipticity constant and the same* L^∞*-bound.*

Our next aim is to introduce the solution concept for the thermistor problem. To this end, we make the following assumption (cf. also Remark 3.25 below):

ASSUMPTION 3.4 (Maximal elliptic regularity). *There is a* $q\in(3,4)$ *such that the mappings*

$$-\nabla\cdot\rho\nabla : W^{1,q}_{\Gamma_D}\to W^{-1,q}_{\Gamma_D} \tag{3.3}$$

and

$$-\nabla\cdot\kappa\nabla+1 : W^{1,q}\to W^{-1,q}_\emptyset \tag{3.4}$$

each provide a topological isomorphism.

The papers [41, Appendix] and [16] provide a zoo of arrangements such that Assumption 3.4 is satisfied. Note that it is not presumptuous to assume that both differential operators provide topological isomorphisms at the same time, since the latter property mainly depends on the behavior of the discontinuous coefficient functions (versus the geometry of Γ_D), and these correspond to the material properties in the workpiece described by the domain Ω, i.e., the coefficient functions should exhibit similar properties with regard to jumps or discontinuities in general, the main obstacles to overcome for the isomorphism property. Since κ is not assumed to be continuous, Assumption (3.4) is not satisfied *a priori*, even though no mixed boundary conditions are present, see [18, Ch. 4] for a striking example. In this sense, mixed boundary conditions are not a stronger obstruction against higher regularity in the range $q \in (3,4)$ than discontinuous coefficient functions are.

REMARK 3.5. *In case of mixed boundary conditions it does not make sense to demand Assumption 3.4—even if all data are smooth—for $q \geq 4$, due to Shamir's famous counterexample [52].*

In order to treat the quasilinearity in (1.1), we need to ensure a certain uniformity of domains of the differential operator $-\nabla \cdot \eta(\theta)\kappa\nabla$ during the evolution. To this end, we first note that the isomorphism-property for $-\nabla \cdot \kappa\nabla + 1$ from Assumption 3.4 extends to a broader class of coefficient functions.

DEFINITION 3.6. *Let $\underline{C}(\overline{\Omega})$ denote the set of positive functions on Ω which are uniformly continuous and admit a positive lower bound.*

LEMMA 3.7. *Assume that Assumption 3.4 holds for some number $q \in [2,4)$. If $\xi \in \underline{C}(\overline{\Omega})$, then (3.3) and (3.4) remain topological isomorphisms, if ρ and κ are replaced by $\xi\rho$ and $\xi\kappa$, respectively.*

A proof can be found in [16, Ch. 6].

COROLLARY 3.8. *Assume that (3.4) is a topological isomorphism for some $q \in [2,4)$. Then, for every $\xi \in \underline{C}(\overline{\Omega})$, the domain of the operator $-\nabla\cdot\xi\kappa\nabla+\tilde{\alpha}$, considered in $W^{-1,q}_{\emptyset}$, is still $W^{1,q}$. In particular, for every function $\zeta \in C(\overline{\Omega})$, the operator $-\nabla \cdot \eta(\zeta)\kappa\nabla + \tilde{\alpha}$ has domain $W^{1,q}$.*

Proof. The first assertion follows from Lemma 3.7 and relative compactness of the boundary integral in $\tilde{\alpha}$ with respect to $-\nabla \cdot \xi\kappa\nabla$, compare [39, Ch. IV.1.3]. For the second assertion, note that η is assumed to be Lipschitzian on bounded intervals and bounded from below by 0 as in Assumption 2.6. Thus, $\eta(\zeta)$ is uniformly continuous and has a strictly positive lower bound. ∎

REMARK 3.9. *Let us briefly recall the considerations from the introduction, regarding a volumetric heat capacity term in the form $\varrho C_p(\theta)$ in front of the time derivative of θ. As explained there, one may use the so-called enthalpy transformation to get rid of the additional dependency on θ, thereby modifying the data η, σ and α. Now, considering that we are allowing κ and ρ to be spatially discontinuous to account for heterogeneous material, one might be tempted to let ϱ also be of that form, say, $\varrho \in L^\infty$ with a strictly positive essential lower bound. However, in order to return to a divergence-gradient structure as in (1.1), this essentially requires ϱ to act as a multiplier on $W^{-1,q}_{\emptyset}$ which calls for $\varrho \in W^{1,q}$ – in particular, L^∞* is not *enough. We refer to [8, Sect. 3], [31, Ch. 6].*

We are now in the position to define what is to be understood as a *solution* to the system (1.1)–(1.6).

DEFINITION 3.10. *We define*

$$\mathcal{A}(\zeta) := -\nabla \cdot \eta(\zeta)\kappa\nabla + \tilde{\alpha}$$

as a mapping $\mathcal{A}\colon C(\overline{\Omega}) \to \mathcal{L}(W^{1,q}; W^{-1,q}_{\emptyset})$.

DEFINITION 3.11. *The number $r^*(q) = \frac{2q}{q-3}$ is called the* critical exponent.

DEFINITION 3.12 (Abstract solution concept). *Let $q > 3$ and let r be from $(r^*(q), \infty)$. We say that the pair (θ, φ) with $\theta(T_0) = \theta_0$ is a* local solution *of the thermistor-problem on $J_{\max} := (T_0, T_{\max})$ for $T_{\max} \in (T_0, T_1]$ if (θ, φ) satisfies the* *equations*

$$\partial_t\theta(t) + \mathcal{A}(\theta(t))\theta(t) = (\sigma(\theta(t))\rho\nabla\varphi(t))\cdot\nabla\varphi(t) + \alpha\theta_l(t) \qquad \text{in } W^{-1,q}_{\emptyset}, \tag{3.5}$$

$$-\nabla\cdot\sigma(\theta(t))\rho\nabla\varphi(t) = u(t) \qquad \text{in } W^{-1,q}_{\Gamma_D} \tag{3.6}$$

for almost all $t \in (T_0, T_{\max})$ *and admits the regularity*

$$\varphi \in L^{2r}(T_0, T_\bullet; W^{1,q}_{\Gamma_D}) \quad \text{and} \quad \theta \in W^{1,r}(T_0, T_\bullet; W^{-1,q}_{\emptyset}) \cap L^r(T_0, T_\bullet; W^{1,q}) \tag{3.7}$$

for every $T_\bullet \in J_{\max}$*. If* (3.7) *is not true for* $T_\bullet = T_{\max}$*, then we say that* (θ,φ) *is a* maximal *local solution and call* $J_{\max}$ *the* maximal interval of existence*. If* $J_{\max} = J$ *and even*

$$\varphi \in L^{2r}(J; W^{1,q}_{\Gamma_D}) \quad \text{and} \quad \theta \in W^{1,r}(J; W^{-1,q}_{\emptyset}) \cap L^r(J; W^{1,q}), \tag{3.8}$$

then we call (θ,φ) *a* global solution*.*

Remark 3.13 (Comments on the solution concept).

(i) *In the context of Definition 3.12,* $\partial_t\theta$ *always means the time derivative of* θ *in the sense of vector-valued distributions, see [1, Ch. III.1] or [21, Ch. IV].*

(ii) *Via* (3.10) *and Corollary 3.20 below, we will see that a solution* θ *in the above sense is in fact Hölder-continuous on* $\overline{\Omega \times J}$*. In particular,* $\theta(t)$ *is uniformly continuous on* Ω *for every* $t \in J$*, such that* $\mathcal{A}(\theta(t))$ *is well-defined according to Definition 3.10.*

(iii) *The reader will verify that the boundary conditions imposed on* φ *in* (1.5) *and* (1.6) *are incorporated in this definition in the spirit of [21, Ch. II.2] or [13, Ch. 1.2]. For an adequate interpretation of the boundary conditions for* θ *as in* (1.2)*, see [45, Ch. 3.3.2] and the in-book references there.*

(iv) *We give a short comment on the so-called Thermistor trick performed by rewriting the Joule heat sources on the right-hand side of* (3.5) *(formally) to*

$$\int_\Omega \big(\varphi\sigma(\theta)\varrho\nabla\varphi\big)\cdot\nabla v \,\mathrm{d}x = \langle -\nabla\cdot\varphi\sigma(\theta)\varrho\nabla\varphi, v\rangle + \int_{\Gamma_N} \varphi\, u\, v \,\mathrm{d}\omega,$$

cf. e.g. [6]. On account of L^∞*-regularity results for elliptic equations, see e.g. [40, Thm. II.B.2], this allows to treat both equations in a Hilbert– resp. energy space setting. We have however decided not to build upon this trick because it is not clear how to solve the resulting quasi-linear equations in the energy spaces. As we will see below, our ansatz strongly depends on higher regularity for* θ *which we are unable to achieve in the Hilbert space setting.*

(v) *The unfortunately rather convoluted definition regarding maximal and global solutions is necessary because of the embedding* (3.9) *below. If* (3.7) *was true also for* $T_\bullet = T_{\max}$*, then one could find a continuation of* (θ,φ) *using the initial value* $\theta(T_{\max})$ *to a larger time interval* $(T_0, T^\bullet_{\max}) \supset (T_0, T_{\max})$ *(cf. the proof of Theorem 3.14). In this sense,* $T_{\max}$ *could not be called "maximal" in this case. The maximal time of existence* $T_{\max}$ *is moreover equivalently characterized by the property that* $\lim_{t\nearrow T_{\max}}\theta(t)$ *does not exist in* $(W^{1,q}, W^{-1,q}_{\emptyset})_{\frac{1}{r},r}$*, see [50, Cor. 3.2].*

We are now going to formulate the main result of this part.

Theorem 3.14 (Existence and uniqueness for local solutions). *Let* $q \in (3,4)$ *be a number for which Assumption 3.4 is satisfied,* $r > r^*(q)$ *and* $u \in L^{2r}(J; W^{-1,q}_{\Gamma_D})$*, where* $r^*(q)$ *is the critical exponent from Definition 3.11. If* θ_0 *is from* $(W^{1,q}, W^{-1,q}_{\emptyset})_{\frac{1}{r},r}$*, then there is a unique* maximal local *solution of* (3.5) *and* (3.6) *in the sense of Definition 3.12.*

The proof of this theorem is given in the next subsection.

3.2. Local existence and uniqueness for the state system: the proof. Let us first briefly sketch the proof of Theorem 3.14 by giving an overview over the steps:

- The overall proof is based on a local existence result of Prüss for abstract quasilinear parabolic equations, whose principal part satisfies a certain maximal parabolic regularity property, see [50] and Proposition 3.17.
- For the application of this abstract result to our problem, we reduce the thermistor system to an equation in the temperature θ only by solving the elliptic equation for φ in dependence of θ. This gives rise to a nonlinear operator S appearing in the reduced equation for θ, see Definition 3.26 and Proposition 3.28.
- The key tool to verify the assumptions on S for the application of Prüss' result is Lemma 3.7, which is the basis for the proof of Lemma 3.27. The application of Lemma 3.7 to obtain Lipschitz-continuity requires to treat the temperature in a space which (compactly) embeds into $C(\overline{\Omega})$. This issue is addressed by Corollary 3.20.

Before we start with the proof itself, let us first recall the concept of maximal parabolic regularity, a crucial tool in the following considerations, and point out some basic facts on this:

Definition 3.15 (Maximal parabolic regularity). *Let X be a Banach space and A be a closed operator with dense domain* $\operatorname{dom}(A) \subset X$. *Suppose* $\mathfrak{r} \in (1, \infty)$. *Then we say that A has* maximal parabolic $L^{\mathfrak{r}}(J;X)$-regularity, *iff for every $f \in L^{\mathfrak{r}}(J;X)$ there is a unique function $w \in W^{1,\mathfrak{r}}(J;X) \cap L^{\mathfrak{r}}(J;\operatorname{dom}(A))$ which satisfies*

$$\partial_t w(t) + Aw(t) = f(t), \quad w(T_0) = 0$$

in X for almost every $t \in J = (T_0, T_1)$.

Remark 3.16 (Known results on maximal parabolic regularity).

(i) *If A satisfies maximal parabolic $L^{\mathfrak{r}}(J;X)$-regularity, then it does so for any other (bounded) time interval, see [17].*

(ii) *If A satisfies maximal parabolic $L^{\mathfrak{r}}(J;X)$-regularity for* some $\mathfrak{r} \in (1,\infty)$, *then it satisfies maximal parabolic $L^{\mathfrak{r}}(J;X)$-regularity for* all $\mathfrak{r} \in (1,\infty)$, *see [53] or [17].*

(iii) *Let Y be another Banach space, being dense in X with $Y \hookrightarrow X$. Then there are the embeddings*

$$W^{1,\mathfrak{r}}(J;X) \cap L^{\mathfrak{r}}(J;Y) \hookrightarrow C(\overline{J};(Y,X)_{\frac{1}{\mathfrak{r}},\mathfrak{r}}) \tag{3.9}$$

and

$$W^{1,\mathfrak{r}}(J;X) \cap L^{\mathfrak{r}}(J;Y) \hookrightarrow C^{\varrho}(\overline{J};(Y,X)_{\zeta,1}) \tag{3.10}$$

where $0 < \varrho \le \zeta - \frac{1}{\mathfrak{r}}$, see [2, Ch. 3, Thm. 3]. In the immediate context of maximal parabolic regularity, Y is taken as $\operatorname{dom}(A)$ *equipped with the graph norm, of course. According to (i) and (ii), we only say that A satisfies* maximal parabolic regularity on X.

In the following, we establish some preliminary results for the proof of Theorem 3.14, which will heavily rest on the following fundamental theorem of Prüss:

Proposition 3.17 (Abstract quasilinear evolution equations [50, Thm. 3.1]). *Let Y, X be Banach spaces, Y dense in X, such that $Y \hookrightarrow X$ and set $J = (T_0, T_1)$ and $\mathfrak{r} \in (1,\infty)$. Suppose that A maps $(Y,X)_{\frac{1}{\mathfrak{r}},\mathfrak{r}}$ into $\mathcal{L}(Y;X)$ such that $A(w_0)$ satisfies maximal parabolic regularity on X with* $\operatorname{dom}(A(w_0)) = Y$ *for some $w_0 \in (Y,X)_{\frac{1}{\mathfrak{r}},\mathfrak{r}}$. Let, in addition, $S\colon J \times (Y,X)_{\frac{1}{\mathfrak{r}},\mathfrak{r}} \to X$ be a Carathéodory map and $S(\cdot,0)$ be from $L^{\mathfrak{r}}(J;X)$. Moreover, let the following two assumptions be satisfied:*

(A) *For every $M > 0$, there is a constant $L(M)$ such that for all $w, \bar{w} \in (Y,X)_{\frac{1}{\mathfrak{r}},\mathfrak{r}}$, where $\max(\|w\|_{(Y,X)_{\frac{1}{\mathfrak{r}},\mathfrak{r}}}, \|\bar{w}\|_{(Y,X)_{\frac{1}{\mathfrak{r}},\mathfrak{r}}}) \le M$, we have*

$$\|A(w) - A(\bar{w})\|_{\mathcal{L}(Y;X)} \le L(M)\|w - \bar{w}\|_{(Y,X)_{\frac{1}{\mathfrak{r}},\mathfrak{r}}}.$$

(S) *For every $M > 0$, assume that there is a function $h_M \in L^{\mathfrak{r}}(J)$ such that for all $w, \bar{w} \in (Y,X)_{\frac{1}{\mathfrak{r}},\mathfrak{r}}$, where $\max(\|w\|_{(Y,X)_{\frac{1}{\mathfrak{r}},\mathfrak{r}}}, \|\bar{w}\|_{(Y,X)_{\frac{1}{\mathfrak{r}},\mathfrak{r}}}) \le M$, it is true that*

$$\|S(t,w) - S(t,\bar{w})\|_X \le h_M(t)\|w - \bar{w}\|_{(Y,X)_{\frac{1}{\mathfrak{r}},\mathfrak{r}}} \tag{3.11}$$

for almost every $t \in J$.

Then, for each $w_0 \in (Y,X)_{\frac{1}{\mathfrak{r}},\mathfrak{r}}$, there exists $T_{\max} \in (T_0, T_1]$ such that the problem

$$\begin{cases} \partial_t w(t) + A(w(t))w(t) = S(t, w(t)) \quad \text{in } X, \\ \qquad\qquad w(T_0) = w_0 \end{cases} \tag{3.12}$$

admits a unique *solution $w \in W^{1,\mathfrak{r}}(T_0, T_\bullet; X) \cap L^{\mathfrak{r}}(T_0, T_\bullet; Y)$ on $(T_0, T_\bullet)$ for every $T_\bullet \in (T_0, T_{\max})$.*

REMARK 3.18. *It is known that the solution of the thermistor problem possibly ceases to exist after finite time in general, cf. [6, Ch. 5] and the references therein. Thus, one has to expect here, in contrast to the two-dimensional case treated in [34], only a local-in-time solution. In this scope, Prüss' theorem will prove to be the adequate instrument.*

As indicated above, we will prove Theorem 3.14 by reducing the thermistor system to an equation in the temperature only and apply Proposition 3.17 to this equation. To be more precise, we first establish the assumptions **(A)** for $\mathfrak{r} = r > r^*(q)$ and $\mathcal{A}$ as defined in Definition 3.10. We then solve the elliptic equation (3.6) for φ (uniquely) for every time point t in dependence of a function ζ and $u(t)$, where ζ enters the equation inside the coefficient function $\sigma(\zeta)\rho$. Then the right-hand side of the parabolic equation (3.5) may be written also as a function S solely of t and ζ. We then show that this function satisfies the suppositions **(S)** in Prüss' theorem.

To carry out this concept, we need several prerequisites: here our first central aim is to show that indeed the mapping $(W^{1,q}, W^{-1,q}_\emptyset)_{\frac{1}{r},r} \ni \zeta \mapsto \mathcal{A}(\zeta)$ from Definition 3.10 satisfies the assumptions from Proposition 3.17 for $r > r^*(q)$, cf. Lemma 3.21 below. For doing so, we first investigate the spaces $(W^{1,q}, W^{-1,q}_\emptyset)_{\zeta,1}$ in view of their embedding into Hölder spaces. For later use, the subsequent result is formulated slightly broader as presently needed. Its proof is postponed to the appendix, see Lemma A.1. Note that we will need the range $\varsigma \in (1,2)$ and thus the volume-preserving property for $\Omega \cup \Gamma_D$ only in §4.

LEMMA 3.19. *Let $q \in (3,4)$ and $\varsigma \in [2,q]$. For every $\tau \in (0, \frac{q-3}{2q}(1-\frac{3}{q}+\frac{3}{\varsigma})^{-1})$, the interpolation space $(W^{1,q}, W^{-1,\varsigma}_\emptyset)_{\tau,1}$ embeds into some Hölder space $C^\delta(\overline{\Omega})$ with $\delta > 0$. If $\Omega \cup \Gamma_D$ even satisfies the volume-conservation condition from Definition 2.3 (ii), then we can additionally admit $\varsigma \in (1,2)$.*

We immediately obtain the following crucial consequences:

COROLLARY 3.20.

(i) *Let $q > 3$ and $\varsigma \in [2,q]$. Then, for every $s > \frac{2q}{q-3}(1 - \frac{3}{q} + \frac{3}{\varsigma})$, the interpolation space $(W^{1,q}, W^{-1,\varsigma}_\emptyset)_{\frac{1}{s},s}$ embeds into some Hölder space $C^\delta(\overline{\Omega})$, and thus even compactly into $C(\overline{\Omega})$. If $\Omega \cup \Gamma_D$ even satisfies the volume-conservation condition from Definition 2.3 (ii), then we can additionally admit $\varsigma \in (1,2)$.*

(ii) *Under the same supposition, there exists a* $\varrho > 0$ *such that*

$$W^{1,s}(J; W_\emptyset^{-1,\varsigma}) \cap L^s(J; W^{1,q}) \hookrightarrow C^\varrho(\overline{J}; C^\varrho(\overline{\Omega})).$$

(iii) *Let Assumption 3.4 hold true for some* $q \in (3,4)$. *Then the operator* $\mathcal{A}(\zeta)$ *satisfies maximal parabolic regularity on* $W_\emptyset^{-1,q}$ *with domain* $W^{1,q}$ *for every* $\zeta \in (W^{1,q}, W_\emptyset^{-1,q})_{\frac{1}{r},r}$ *with* $r > r^*(q)$, *where* $r^*(q)$ *is the critical exponent from Definition 3.11.*

Proof. (i) We have $(W^{1,q}, W_\emptyset^{-1,\varsigma})_{\frac{1}{s},s} \hookrightarrow (W^{1,q}, W_\emptyset^{-1,\varsigma})_{\iota,1}$ for every $\iota \in (\frac{1}{s}, 1)$. The condition on s implies that the interval $\mathcal{I} := (\frac{1}{s}, \frac{q-3}{2q}(1 - \frac{3}{q} + \frac{3}{\varsigma})^{-1})$ is nonempty. Taking ι from $\mathcal{I}$, the assertion follows from Lemma 3.19. (ii) follows from Lemma 3.19 and Remark 3.16. (iii) The claim follows from uniform continuity of functions from $(W^{1,q}, W_\emptyset^{-1,q})_{\frac{1}{r},r}$ by (i), Lemma 3.7 for $\xi := \eta(\zeta)$ and [7, Thm. 11.5] using the technique as in [31, Thm. 5.16/Lem. 5.15]. ∎

Setting $\varsigma = q$ in Corollary 3.20 (i) and (ii) gives the condition $r > r^*(q) = \frac{2q}{q-3}$ for the assertions to hold with $s = r$. We will use this special case frequently in the course of the remaining part of this section. Let us now turn to the operator $\mathcal{A}$.

PROPOSITION 3.21. *Suppose that Assumption 3.4 holds true for some* $q \in (3,4)$ *and that* $\theta_0 \in (W^{1,q}, W_\emptyset^{-1,q})_{\frac{1}{r},r}$ *where* $r > r^*(q)$. *With* $\mathcal{A}$ *as in Definition 3.10, the function* $(W^{1,q}, W_\emptyset^{-1,q})_{\frac{1}{r},r} \ni \zeta \mapsto \mathcal{A}(\zeta)$ *then satisfies the assumptions from Proposition 3.17 for the spaces* $X = W_\emptyset^{-1,q}$ *and* $Y = W^{1,q}$.

Proof. With $\varsigma = q$, Corollary 3.20 shows that $(W^{1,q}, W_\emptyset^{-1,q})_{\frac{1}{r},r} \hookrightarrow C(\overline{\Omega})$, such that the operator $\mathcal{A}$ indeed maps $(W^{1,q}, W_\emptyset^{-1,q})_{\frac{1}{r},r}$ into $\mathcal{L}(W^{1,q}; W_\emptyset^{-1,q})$ by Corollary 3.8. Using Lipschitz continuity of η on bounded sets and Remark 3.2, we also obtain **(A)**: Let $w, \bar{w} \in (W^{1,q}, W_\emptyset^{-1,q})_{\frac{1}{r},r}$ with norms bounded by $M > 0$. Then we have

$$\begin{aligned}
\|\mathcal{A}(w) - \mathcal{A}(\bar{w})\|_{\mathcal{L}(W^{1,q}; W_\emptyset^{-1,q})} &= \|\nabla \cdot (\eta(w) - \eta(\bar{w}))\,\kappa\nabla\|_{\mathcal{L}(W^{1,q}, W_\emptyset^{-1,q})} \\
&\le L_\eta \|\kappa\|_{L^\infty} \|w - \bar{w}\|_{C(\overline{\Omega})} \\
&\le C L_\eta \|\kappa\|_{L^\infty} \|w - \bar{w}\|_{(W^{1,q}, W_\emptyset^{-1,q})_{\frac{1}{r},r}}.
\end{aligned}$$

Finally, the property of maximal parabolic regularity for $\mathcal{A}(\theta_0)$ follows immediately from Corollary 3.20. ∎

Next we will establish and investigate the right hand hand side of (3.12). For doing so, we now turn our attention to the elliptic equation (3.6).

LEMMA 3.22. *For* $q \ge 2$ *and* $\zeta \in C(\overline{\Omega})$, $\mathfrak{a}_\zeta(\varphi_1, \varphi_2) := (\sigma(\zeta)\rho\nabla\varphi_1) \cdot \nabla\varphi_2$ *defines a continuous bilinear form* $\mathfrak{a}_\zeta \colon W^{1,q}_{\Gamma_D} \times W^{1,q}_{\Gamma_D} \to L^{q/2}$. *Moreover,* $(\zeta, \varphi) \mapsto \mathfrak{a}_\zeta(\varphi, \varphi)$ *is Lipschitzian over bounded sets in* $C(\overline{\Omega}) \times W^{1,q}_{\Gamma_D}$.

Proof. Bilinearity and continuity of each $\mathfrak{a}_\zeta$ are clear. The second assertion follows from a straightforward calculation with the resulting estimate

$$\begin{aligned}
\|\mathfrak{a}_{\zeta_1}(\varphi_1, \varphi_1) - \mathfrak{a}_{\zeta_2}(\varphi_2, \varphi_2)\|_{L^{q/2}} &\le \|\sigma(\zeta_1) - \sigma(\zeta_2)\|_{L^\infty} \|\rho\|_{L^\infty} \|\varphi_1\|^2_{W^{1,q}_{\Gamma_D}} \\
&\quad + 2\|\sigma(\zeta_2)\|_{L^\infty} \|\rho\|_{L^\infty} \|\varphi_1\|_{W^{1,q}_{\Gamma_D}} \|\varphi_1 - \varphi_2\|_{W^{1,q}_{\Gamma_D}},
\end{aligned}$$

Lipschitz continuity of σ and boundedness of the underlying sets. ∎

Let us draw some further conclusions from Lemma 3.7. For this, we assume Assumption 3.4 for the rest of this chapter.

Lemma 3.23. *The mapping*

$$\underline{C}(\overline{\Omega}) \ni \phi \mapsto (-\nabla\cdot\phi\rho\nabla)^{-1} \in \mathcal{LH}(W^{-1,q}_{\Gamma_D}; W^{1,q}_{\Gamma_D})$$

is well-defined and even continuous.

Proof. The well-definedness assertion results from Lemma 3.7. The second assertion is implied by the first, Remark 3.2 and the continuity of the mapping $\mathcal{LH}(X;Y) \ni B \mapsto B^{-1} \in \mathcal{LH}(Y;X)$, see [51, Ch. III.8]. ∎

Corollary 3.24. *Let $\underline{\mathfrak{C}} \subset \underline{C}(\overline{\Omega})$ be a compact set in $C(\overline{\Omega})$ which admits a common lower positive bound. Then the function*

$$\underline{\mathfrak{C}} \ni \phi \mapsto \mathcal{J}(\phi) := (-\nabla\cdot\phi\rho\nabla)^{-1} \in \mathcal{LH}(W^{-1,q}_{\Gamma_D}; W^{1,q}_{\Gamma_D})$$

is bounded and even Lipschitzian. The same holds for $\underline{\mathfrak{C}}\times\mathfrak{B} \ni (\phi,v) \mapsto \mathcal{J}(\phi)v \in W^{1,q}_{\Gamma_D}$ for every bounded set $\mathfrak{B} \subset W^{-1,q}_{\Gamma_D}$.

Proof. Lemma 3.23 and the compactness of $\underline{\mathfrak{C}}$ in $C(\overline{\Omega})$ immediately imply boundedness of $\mathcal{J}$ on $\underline{\mathfrak{C}}$. In turn, Lipschitz continuity of $\mathcal{J}$ is obtained from boundedness and the resolvent-type equation

$$\begin{aligned}(-\nabla\cdot\phi_1\rho\nabla)^{-1} &- (-\nabla\cdot\phi_2\rho\nabla)^{-1}\\ &= (-\nabla\cdot\phi_1\rho\nabla)^{-1}(-\nabla\cdot(\phi_2-\phi_1)\rho\nabla)(-\nabla\cdot\phi_2\rho\nabla)^{-1}\end{aligned}$$

(read: $A^{-1}-B^{-1} = A^{-1}(B-A)B^{-1}$) and Remark 3.2. Considering the assertion on the combined mapping, boundedness is obvious and further we have for $\phi_1,\phi_2 \in \underline{\mathfrak{C}}$ and $v_1,v_2 \in \mathfrak{B}$:

$$\begin{aligned}\|\mathcal{J}(\phi_1)v_1 - \mathcal{J}(\phi_2)v_2\|_{W^{1,q}_{\Gamma_D}} &\le \|\mathcal{J}(\phi_1)-\mathcal{J}(\phi_2)\|_{\mathcal{L}(W^{-1,q}_{\Gamma_D},W^{1,q}_{\Gamma_D})}\,\|v_1\|_{W^{-1,q}_{\Gamma_D}}\\ &\quad + \|\mathcal{J}(\phi_2)\|_{\mathcal{L}(W^{-1,q}_{\Gamma_D},W^{1,q}_{\Gamma_D})}\|v_1-v_2\|_{W^{-1,q}_{\Gamma_D}}.\end{aligned}$$

With Lipschitz continuity and boundedness of $\mathcal{J}$ over $\underline{\mathfrak{C}}$ and boundedness of $\mathfrak{B}$, this implies the claim. ∎

Remark 3.25. *At this point we are in the position to discuss the meaning of Assumption 3.4 in some detail. Under Assumption 2.5 (i) for a closed subset Ξ of $\partial\Omega$, it is known that, even for arbitrary measurable, bounded, elliptic coefficient functions μ, $(\mathcal{D}_q(\mu), W^{-1,q}_{\Xi})_{\tau,1}$ embeds into a Hölder space for suitable τ, cf. [32, Cor. 3.7] (for $\mathcal{D}_q(\mu)$, see Remark 3.2). In particular, one does* not *need an assumption for the isomorphism property between $W^{1,q}_{\Xi}$ and $W^{-1,q}_{\Xi}$ for this result. The crucial point behind Assumption 3.4 is to achieve both independence of the domains for the operators $-\nabla\phi\mu\nabla$ within a suitable class of functions ϕ, as well as a well-behaved dependence on ϕ in the space $\mathcal{L}(\mathcal{D}_q; W^{-1,q}_{\Xi})$, cf. Lemma 3.7 and Corollaries 3.8 and 3.24.*

The next lemmata establish the right-hand side in (3.12) with the correct regularity and properties. Moreover, Lipschitz continuity with respect to the control u in the elliptic equation is shown along the way, which will become useful in later considerations. Recall that $\sigma\colon \mathbb{R} \to (0,\infty)$ is Lipschitzian on any finite interval by Assumption 2.6.

Definition 3.26. *We assign to $\zeta \in C(\overline{\Omega})$ and $v \in W^{-1,q}_{\Gamma_D}$ the solution φ_v of $-\nabla\cdot\sigma(\zeta)\rho\nabla\varphi_v = v$ via $\varphi_v = \mathcal{J}(\sigma(\zeta))v$ with $\mathcal{J}$ as in Corollary 3.24. Moreover, set*

$$\Psi_v(\zeta) := \mathfrak{a}_\zeta(\mathcal{J}(\sigma(\zeta))v, \mathcal{J}(\sigma(\zeta))v)$$

for $\zeta \in C(\overline{\Omega})$ *with* $\mathfrak{a}_\zeta$ *as in Lemma 3.22.*

LEMMA 3.27. *Let* $\mathfrak{C}$ *be a compact subset of* $C(\overline{\Omega})$ *and* $\mathfrak{B}$ *a bounded set in* $W^{-1,q}_{\Gamma_D}$*. Then* $(v,\zeta) \mapsto \Psi_v(\zeta)$ *is Lipschitzian from* $\mathfrak{B} \times \mathfrak{C}$ *into* $L^{q/2}$ *and the Lipschitz constant of* $\zeta \mapsto \Psi_v(\zeta)$ *is bounded over* $v \in \mathfrak{B}$*.*

Proof. For every $\zeta \in \mathfrak{C}$, the function $\sigma(\zeta)$ belongs to $\underline{C}(\overline{\Omega})$, thus $\mathcal{J}(\sigma(\zeta))v$ is indeed from $W^{1,q}_{\Gamma_D}$ thanks to Lemma 3.7. Hence, $\Psi_v(\zeta) \in L^{q/2}$ is clear by Hölder's inequality. Let us show the Lipschitz property of Ψ: First, note that Nemytskii operators induced by Lipschitz functions preserve compactness in the space of continuous functions, and note further that the set of all $\sigma(\zeta)$ for $\zeta \in \mathfrak{C}$ admits a common positive lower bound by the Lipschitz property of σ. Hence, the set $\{\sigma(\zeta) \colon \zeta \in \mathfrak{C}\}$ satisfies the assumptions in Lemma 3.22 and Corollary 3.24. For $\zeta_1, \zeta_2 \in \mathfrak{C}$ and $v_1, v_2 \in W^{-1,q}_{\Gamma_D}$, we first obtain via Lemma 3.22

$$\|\Psi_{v_1}(\zeta_1) - \Psi_{v_2}(\zeta_2)\|_{L^{q/2}} \leq L_{\mathfrak{a}} \left(\|\zeta_1 - \zeta_2\|_{C(\overline{\Omega})} + \|\mathcal{J}(\sigma(\zeta_1))v_1 - \mathcal{J}(\sigma(\zeta_2))v_2\|_{W^{1,q}_{\Gamma_D}} \right)$$

and further with Corollary 3.24

$$\|\mathcal{J}(\sigma(\zeta_1))v_1 - \mathcal{J}(\sigma(\zeta_2))v_2\|_{W^{1,q}_{\Gamma_D}} \leq L_{\mathcal{J}} \left(\|\sigma(\zeta_1) - \sigma(\zeta_2)\|_{C(\overline{\Omega})} + \|v_1 - v_2\|_{W^{-1,q}_{\Gamma_D}} \right).$$

The assertion follows since σ was Lipschitz continuous. Uniformity of the Lipschitz constant of $\zeta \mapsto \Psi_v(\zeta)$ is immediate from the previous considerations. ∎

Following the strategy outlined above, we will specify the mapping S from Proposition 3.17 for our case and show that it satisfies the required conditions.

PROPOSITION 3.28. *Let* $q \in (3,4)$ *be such that Assumption 3.4 is satisfied,* $r > r^*(q)$*, and* $u \in L^{2r}(J; W^{-1,q}_{\Gamma_D})$*. We set*

$$S(t,\zeta) := \Psi_{u(t)}(\zeta) + \alpha\theta_l(t).$$

Then S *satisfies the conditions from Proposition 3.17 for the spaces* $X = W^{-1,q}_{\emptyset}$ *and* $Y = W^{1,q}$*.*

Proof. We show that $S(\cdot,0) \in L^r(J; W^{-1,q}_{\emptyset})$. The function $\alpha\theta_l$ is essentially bounded in time with values in $W^{-1,q}_{\Gamma_D}$ by virtue of Remark 2.7 and thus poses no problem here. For almost all $t \in J$, we further have

$$\left\|\Psi_{u(t)}(0)\right\|_{L^{q/2}} \leq |\sigma(0)| \|\rho\|_{L^\infty} \|\mathcal{J}(\sigma(0))\|^2_{\mathcal{L}(W^{-1,q}_{\Gamma_D}; W^{1,q}_{\Gamma_D})} \|u(t)\|^2_{W^{-1,q}_{\Gamma_D}}.$$

Since u is $2r$-integrable in time, this means that $\Psi_{u(t)}(0) \in L^r(J; L^{q/2})$. Due to $q > 3$ and thus $L^{q/2} \hookrightarrow W^{-1,q}_{\emptyset}$ (cf. Remark 2.2), we hence have $S(\cdot,0) \in L^r(J; W^{-1,q}_{\emptyset})$. Let us now show the Lipschitz condition (3.11). If $\mathfrak{C} \subset (W^{1,q}, W^{-1,q}_{\emptyset})_{\frac{1}{r},r}$ is bounded, its closure $\overline{\mathfrak{C}}$ with respect to the sup-norm on $\overline{\Omega}$ forms a compact set in $C(\overline{\Omega})$ by Corollary 3.20. The desired Lipschitz estimate for $S(t,\cdot)$ now follows immediately from Lemma 3.27. ∎

Note that this is the point where the supposition on the time-integrability of u from Assumption 2.6 (vi) comes into play. Essentially, $\Psi_{u(t)}(\zeta)$ only admits half the time-integrability of u, but Propositions 3.21 and 3.28 both require $r > r^*(q)$ to make use of the (compact) embedding $(W^{1,q}, W^{-1,q}_{\emptyset})_{\frac{1}{r},r} \hookrightarrow C(\overline{\Omega})$. Hence, we need more than $2r^*(q)$-integrability for u in time.

Now we have established all ingredients to prove Theorem 3.14. For this purpose, let the assumptions of Theorem 3.14 hold.

Proof of Theorem 3.14. Combining Propositions 3.21 and 3.28 with Proposition 3.17, we obtain a local-in-time solution θ of the equation

$$\partial_t\theta(t) + \mathcal{A}(\theta(t))\theta(t) = S(t,\theta(t)), \quad \theta(T_0) = \theta_0$$

on (T_0, T_*) with $T_* \in (T_0, T_1]$, such that

$$\theta \in W^{1,r}(T_0, T_*; W^{-1,q}_\emptyset) \cap L^r(T_0, T_*; W^{1,q}) \hookrightarrow C([T_0, T_*]; (W^{1,q}, W^{-1,q}_\emptyset)_{\frac{1}{r},r})$$

cf. (3.9). If $T_* < T_1$, we may apply Proposition 3.17 again on the interval (T_*, T_1) with initial value $\theta(T_*) \in (W^{1,q}, W^{-1,q}_\emptyset)_{\frac{1}{r},r}$, thus obtaining another local solution on a subinterval of (T_*, T_1), "glue" the solutions together and start again (note that $\mathcal{A}(\theta(t))$ again satisfies maximal parabolic regularity for every $t \in [T_*, T_1)$ by Corollary 3.20). Proceeding this way, we either obtain a maximal local or a global solution in the sense of Definition 3.12 on the maximal interval of existence $J_{\max}$, which is unique in any case and the so-obtained solutions satisfy the correct regularity, cf. [3, Sect. 7].

Now consider $T_\bullet \in J_{\max}$. We now *define* the function $\varphi(t)$ for each $t \in (T_0, T_\bullet)$ as the solution of $-\nabla \cdot \sigma(\theta(t))\rho\nabla\varphi = u(t)$, that is,

$$\varphi(t) := \mathcal{J}(\sigma(\theta(t)))u(t). \tag{3.13}$$

Then φ indeed belongs to $L^{2r}(T_0, T_\bullet; W^{1,q}_{\Gamma_D})$, since $\mathcal{J}(\sigma(\theta(t))$ is uniformly bounded in $\mathcal{L}(W^{-1,q}_{\Gamma_D}; W^{1,q}_{\Gamma_D})$ over $[T_0, T_\bullet]$ due to the compactness of the set $\{\theta(t) \colon t \in [T_0, T_\bullet]\}$ in $C(\overline{\Omega})$ (cf. Corollary 3.20 and Corollary 3.24), and u was from $L^{2r}(J; W^{-1,q}_{\Gamma_D})$.

Obviously, (θ, φ) is then a solution of the thermistor-problem on $(T_0, T_\bullet)$ in the spirit of Definition 3.12 as claimed in Theorem 3.14. □

3.3. Justification of the chosen setting. We end this chapter with some explanations why the chosen setting in spaces of the kind $W^{-1,q}_\emptyset$ and $W^{-1,q}_{\Gamma_D}$ with $q > 3$ is adequate for the problem under consideration.

Let us inspect the requirements on the spaces in which the equations are formulated. Clearly, they need to contain Lebesgue spaces on Ω as well as on the boundary Γ (or on a subset of the boundary like Γ_N), in order to incorporate the inhomogeneous Neumann boundary data present in both equations. The boundary conditions should be reflected by the formulation of the equations in an adequate way, cf. Remark 3.13 (iii). These demands already strongly prejudice spaces of type $W^{-1,q_p}_\emptyset$ for the parabolic equation and $W^{-1,q_e}_{\Gamma_D}$ for the elliptic equation with probably different integrability orders q_p and q_e for each equation. Finally, in order to treat the nonlinear parabolic equation, we need maximal parabolic regularity for the second order divergence operators $\mathcal{A}(\zeta)$ over $W^{-1,q_p}_\emptyset$, which is generally available by Corollary 3.20 (iii) or [31, Thm. 5.16/Rem. 5.14] in a general context.

Further, aiming at continuous solutions θ, which are needed for having fulfillable Constraint Qualifications for (P) in the presence of state constraints, it is necessary that the domain $\mathcal{D}_{q_p}(\sigma(\zeta)\kappa)$ of the differential operators $\mathcal{A}(\zeta)$, cf. Remark 3.2, embeds into the space of continuous functions on $\overline{Q}$. But it is known that solutions y to equations $-\nabla \cdot \mu\nabla y = f$ for $\mu \in L^\infty(\Omega, \mathbb{R}^{n\times n}_{\mathrm{sym}})$ elliptic with $f \in W^{-1,n}_\emptyset$, where n denotes the space dimension, may in general even be unbounded, see [42, Ch. 1.2]. On the other hand, $\mathcal{D}_{q_p}(\sigma(\zeta)\kappa)$ embeds into a Hölder space if $q_p > 3$, see Remark 3.2. These two facts make the requirement $q_p > n = 3$ expedient. Let us now assume that the elliptic equation admits solutions whose gradient is integrable up to some order q_g. Then the right hand side in the parabolic equation prescribes $q_g \geq \frac{6q_p}{q_p+3}$ in order to have the embedding $L^{q_g/2} \hookrightarrow W^{-1,q_p}_{\emptyset}$. From the requirement $q_p > 3$ then follows $q_g > 3$ as well, i.e., the elliptic equation must admit $W^{1,q_g}_{\Gamma_D}$-solutions with $q_g > 3$. With right-hand sides in $W^{-1,q_e}_{\Gamma_D}$, the best possible constellation is thus $q_e = q_g > 3$ again. Having q_e and q_p both in the same range, we simply choose $q = q_e = q_p > 3$.

Moreover, in order to actually have $W^{1,q}_{\Gamma_D}$-solutions to the elliptic equations for all right-hand sides from $W^{-1,q}_{\Gamma_D}$, the operator $-\nabla \cdot \sigma(\zeta)\rho\nabla$ must be a topological isomorphism between $W^{1,q}_{\Gamma_D}$ and $W^{-1,q}_{\Gamma_D}$. It is also a well-established fact that solutions to elliptic equations with bounded and coercive, but discontinuous coefficient functions may admit almost arbitrarily poor integrability properties for gradients of their solutions, see [49] and [18, Ch. 4]. Under Assumption 3.4, we know that this is not the case for $-\nabla \cdot \rho\nabla$ over $W^{-1,q}_{\Gamma_D}$, but it is clear that it is practically impossible to guarantee this also for the operators $-\nabla \cdot \sigma(\zeta)\rho\nabla$ for all ζ, if $\sigma(\zeta)$ is discontinuous in general. However, from Lemma 3.7 we know that if $\sigma(\zeta)$ if uniformly continuous on Ω, then the isomorphism property carries over. This shows that continuous solutions for the parabolic equation are also needed purely from an analytical point of view, without the considerations coming from the optimal control problem, and also explains why Assumption 3.4 is, in a sense, a “minimal” assumption.

4. Global solutions and existence of optimal controls. Our aim in the following section is to establish existence of optimal solutions for (P) coming from the set of control functions which admit a solution on the *whole* time interval. These control functions will be called “global controls”, see Definition 4.1. In view of the state constraints and the end time observation in the objective of (P), it is natural to restrict the optimal control problem to the set of global controls. We will see below that this set is in fact nonempty, and in a companion paper [47], we even show that it is open. The latter property is, however, not needed here in order to show that optimal solutions to (P) exist.

Let us give a brief roadmap for the upcoming considerations. We first establish the notion of a *global* control and show the set of global controls is in fact nonempty since it includes the zero control. Then, we turn to the existence of optimal controls. The arguments follow the classical direct method of the calculus of variations, see Theorem 4.14. To this end, we need essentially two “special” ingredients, as announced in the introduction:

(i) A closedness result for the set of global controls to make sure that the limit of a sequence of global controls is still a global one. Such a result is given in the form of Theorem 4.7, and requires a certain boundedness of the gradient of the temperatures which is ensured by the second addend in the objective in (P).

(ii) A compactness result for the controls under consideration in order to pass to the limit in the nonlinear state system. We choose to consider a stronger space of controls for this, cf. (4.4), induced by the third term in the objective functional, and show that this space compactly embeds into $L^{2r}(J; W^{-1,q}_{\Gamma_D})$ in Lemma 4.12.

The setting and results of §3 are assumed as given. In particular, we consider the assumptions of Theorem 3.14 to be fulfilled and fixed, that means, $q > 3$ such that Assumption 3.4 is satisfied and $r > r^*(q)$ are given from now on, cf. Definition 3.11. For each $u \in L^{2r}(J; W^{-1,q}_{\Gamma_D})$, there exists a maximal local solution (θ_u, φ_u) such that $\theta_u \in W^{1,r}(T_0, T_\bullet; W^{-1,q}) \cap L^r(T_0, T_\bullet; W^{1,q}_{\emptyset})$ for every $T_\bullet \in J_{\max}(u)$, the maximal interval of existence associated to u. We consider $\varphi_u \in L^{2r}(T_0, T_\bullet; W^{1,q}_{\Gamma_D})$ to be given in dependence of u and θ_u as in (3.13). Due to $q > 3$ and $r > r^*(q)$, each solution θ_u is Hölder-continuous on $[T_0, T_\bullet] \times \overline{\Omega}$, cf. Corollary 3.20 (ii). In order to follow the roadmap above, we will use various results from §3 for the solutions (θ_u, φ_u), in particular Proposition 3.3 for uniform bounds for φ_u, albeit in a weaker space, and the continuity properties for the right-hand side $(\theta_u, u) \mapsto \Psi_u(\theta_u)$, culminating in Lemma 3.27. The critical property, via Corollary 3.8, will be continuity of θ_u for each u as established in Corollary 3.20. As already announced, we from now on focus on controls admitting a global solution in the sense of Definition 3.12:

DEFINITION 4.1 (Global controls). *We call a control $u \in L^{2r}(J; W^{-1,q}_{\Gamma_D})$, $r > r^*(q)$, a* global control *if the corresponding solution (θ_u, φ_u) is a global solution in the sense of Definition 3.12, and we denote the set of global controls by $\mathcal{U}_g$. Moreover, we define the* control-to-state operator

$$\mathcal{S} \colon \mathcal{U}_g \ni u \mapsto \mathcal{S}(u) = \theta_u \in W^{1,r}(J; W^{-1,q}_{\emptyset}) \cap L^r(J; W^{1,q})$$

on $\mathcal{U}_g$.

We briefly recall the situation for a global control u: for each $u \in \mathcal{U}_g$, the maximal local solution (θ_u, φ_u) even satisfies $\theta_u \in W^{1,r}(J; W^{-1,q}) \cap L^r(J; W^{1,q}_{\emptyset})$ and accordingly $\varphi_u \in L^{2r}(J; W^{1,q}_{\Gamma_D})$ via (3.13). Again due to the choice of q and r, each such solution θ_u is Hölder-continuous on $\overline{Q}$.

Let us firstly show that the previous definition is in fact meaningful in the sense that $\mathcal{U}_g \neq \emptyset$. The natural candidate for a global control is $u \equiv 0$. One readily observes that the control $u \equiv 0$ leads to the solution $\varphi \equiv 0$ for the elliptic equation (3.6), hence the right-hand side in the parabolic equation reduces to $\alpha\theta_l(t)$ in this case. Indeed, we will show that there exists a *global* solution $\theta_{u\equiv 0}$ to the equation

$$\partial_t \theta(t) + \mathcal{A}(\theta(t))\theta(t) = \alpha\theta_l(t), \quad \theta(T_0) = \theta_0. \tag{4.1}$$

In order to obtain a global solution to (4.1), but also for the proof of existence of optimal controls later in Theorem 4.14, we need the volume-conservation condition which we active at this point for the rest of this paper:

ASSUMPTION 4.2. *In addition to Assumption 2.5, we from now on require that $\Omega \cup \Gamma_D$ satisfies the volume-conservation condition from Definition 2.3 (ii).*

Under this additional assumption, the following result has been shown in [48]. Note that our assumption of regular $\Omega \cup \Gamma_D$ is only a special case of the admissible geometries in [48].

PROPOSITION 4.3 (Global existence for quasilinear equations [48, Thm. 5.3]). *Assume that $\Omega \cup \Xi$ is regular with $\Xi \neq \emptyset$ and in addition satisfies the volume-conservation condition. Let μ be a coefficient function on Ω, measurable, bounded, elliptic. Assume that $\phi : \mathbb{R} \to (0, \infty)$ is Lipschitz continuous on bounded sets. Suppose further that*

$$-\nabla \cdot \mu \nabla : W^{1,q}_{\Xi} \to W^{-1,q}_{\Xi}$$

is a topological isomorphism for some $q > 3$. Let w_0 be from $(W^{1,q}_{\Xi}, W^{-1,q}_{\Xi})_{\frac{1}{r},r}$ with $r > r^*(q) = \frac{2q}{q-3}$. Then, for every $f \in L^r(J; W^{-1,q}_{\Xi})$, there exists a unique global *solution w of the quasilinear equation*

$$\partial_t w(t) - \nabla \cdot \phi(w(t))\mu\nabla w(t) = f(t), \quad w(T_0) = w_0$$

on J, which belongs to $W^{1,r}(J; W^{-1,q}_{\Xi}) \cap L^r(J; W^{1,q}_{\Xi})$.

With $w_0 = \theta_0$, $\Xi = \emptyset$, $\phi = \eta$, $\mu = \kappa$ and $f = \alpha\theta_l$, we may use Proposition 4.3 to ensure the existence of a *global* solution of (4.1) in the sense of Definition 3.12 under

Assumption 3.4 – in particular, $0 \in \mathcal{U}_g$ follows. We summarize these considerations in the following

Corollary 4.4 (Existence of a global control). *The zero control $u \equiv 0$ is a global one, that is, $0 \in \mathcal{U}_g$.*

Remark 4.5. *In [48], Proposition 4.3 is proven for the case where the differential operator consists of the divergence-gradient operator only. However, it is clear that the result extends to the operators of the form $\mathcal{A}$ including the boundary form since the latter is relatively compact with respect to the main part, cf. Corollary 3.8 and the reference there, see also [31, Lem. 5.15].*

Let us turn to the question of existence of an optimal control of (P). Following the standard direct method of the calculus of variations, one soon encounters the problem of lacking uniform boundedness in a suitable space for solutions (θ_{u_n}) associated to a minimizing sequence of global controls (u_n), which is a common obstacle to overcome when treating quasilinear equations. To circumvent this, we use Proposition 3.3 to show that the solutions (θ_{u_n}), in this scenario, are uniformly bounded in $W^{1,s}(J; W^{-1,\varsigma}_\emptyset)$, where $\varsigma \leq 3 < q$ (in general only $\varsigma \sim \frac{3}{2}$) and s is the exponent from the second addend in the objective function in (P). As this term in the objective, together with the state constraints posed in (P), gives an additional bound in $L^s(J; W^{1,q})$, we can employ Corollary 3.20 to "lift" this boundedness result to a Hölder space, which is suitable for passing to the limit with a minimizing sequence. Since $\varsigma < 2$ in general, we need the additional volume-preserving property for $\Omega \cup \Gamma_D$ to be able to use Corollary 3.20 for the range $\varsigma \in (1, 2)$. Moreover, in order to apply Corollary 3.20, the exponent s has to be sufficiently large. The precise bound for s is characterized by the following

Definition 4.6. *Let $\mathfrak{q} \in (2, \min\{q_0, 3\}]$ be given, where q_0 is the number from Proposition 3.3, and set $\varsigma := \frac{3\mathfrak{q}}{6-\mathfrak{q}}$. Then we define the number $\bar{r}(q, \varsigma) > 0$ by*

$$\bar{r}(q, \varsigma) := \frac{2q}{q-3}\left(1 - \frac{3}{q} + \frac{3}{\varsigma}\right).$$

On account of $\varsigma \leq 3 < q$ it follows that $\bar{r}(q, \varsigma) > r^*(q) = \frac{2q}{q-3}$. Therefore, for a given number $s > \bar{r}(q, \varsigma)$, the previous results, in particular the assertions of Theorem 3.14 and Corollary 4.4 hold with $r = s$. The next theorem precisely elaborates the argument depicted before Definition 4.6:

Theorem 4.7 (Closedness properties of $\mathcal{U}_g$). *Let $s > \bar{r}(q, \varsigma)$.*

(i) *Consider a sequence $\mathcal{U}_g \supset (u_n)$ which converges to some $\bar{u} \in L^{2s}(J; W^{-1,q}_{\Gamma_D})$. If the associated sequence of solutions (θ_{u_n}) admits a subsequence which converges to some $\bar{\theta}$ in $C(\overline{Q})$, then $\bar{u} \in \mathcal{U}_g$ and $\bar{\theta} = \theta_{\bar{u}}$.*

(ii) *Let $\mathcal{U} \subseteq \mathcal{U}_g$ be bounded in $L^{2s}(J; W^{-1,q}_{\Gamma_D})$ and suppose in addition that the associated set of solutions $\mathcal{K} = \{\theta_u : u \in \mathcal{U}\}$ is bounded in $L^s(J; W^{1,q})$. Then $\mathcal{K}$ is even compact in $C(\overline{Q})$ and the closure of $\mathcal{U}$ in $L^{2s}(J; W^{-1,q}_{\Gamma_D})$ is still contained in $\mathcal{U}_g$.*

As indicated above, the second addend in the objective functional together with the state constraints will guarantee the bound in $L^s(J; W^{1,q})$ for the minimizing sequence, see the proof of Theorem 4.14 below. Recall the operator $\Psi_u(\theta)$ given by the right hand side in (3.5) in dependence of u, cf. Definition 3.26.

Proof of Theorem 4.7.

(i) For the first assertion, consider the sequence (u_n) from the assumptions with the associated states $(\theta_n) := (\theta_{u_n})$. By assumption, there exists a subsequence of (θ_n), called (θ_{n_k}), which converges to some $\bar{\theta}$ in $C(\overline{Q})$. Lemma 3.27 shows that

$\Psi_{u_{n_k}}(\theta_{n_k}) \to \Psi_{\bar{u}}(\bar{\theta})$ in $L^s(J;W_\emptyset^{-1,q})$ as $k \to \infty$. By [48, Lem. 5.5], the equations

$$\partial_t \zeta + \mathcal{A}(\theta_{n_k})\zeta = \Psi_{u_{n_k}}(\theta_{n_k}) + \alpha\theta_l, \quad \theta_{n_k}(T_0) = \theta_0$$

have unique solutions $\zeta_{n_k} \in W^{1,s}(J;W_\emptyset^{-1,q}) \cap L^s(J;W^{1,q})$, which, due to uniqueness of solutions for the nonlinear state system, must coincide with θ_{n_k}. This means, on the one hand, that $\zeta_{n_k} = \theta_{n_k} \to \bar{\theta}$ in $C(\overline{Q})$ as $k \to \infty$. On the other hand, [48, Lem. 5.5] also shows that the sequence (ζ_{n_k}) has a limit $\bar{\zeta}$ in the maximal regularity space as k goes to infinity, where $\bar{\zeta}$ is the solution of the limiting problem

$$\partial_t \zeta + \mathcal{A}(\bar{\theta})\zeta = \Psi_{\bar{u}}(\bar{\theta}) + \alpha\theta_l, \quad \zeta(T_0) = \theta_0.$$

We do, however, already know that $\bar{\zeta} = \bar{\theta}$, such that $\bar{\theta}$ is the unique global solution to the nonlinear problem for the limiting control $\bar{u}$, i.e., $\bar{\zeta} = \bar{\theta} =: \theta_{\bar{u}}$. In particular, $\bar{u}$ is still a global control. Note that, as explained in Remark 4.5, one needs to extend the result from [48] to the actual operator $\mathcal{A}$, as we consider here, in a straight-forward way.

(ii) We show that $\mathcal{K}$ is bounded in a suitable maximal-regularity-like space. To this end, we first investigate the right-hand side in the parabolic equation (3.5). Denote by (θ_u, φ_u) the solution for a given $u \in \mathcal{U}$. Thanks to Assumption 2.6 (i), Proposition 3.3 shows that, with $\mathfrak{q}$ as in Definition 4.6, $-\nabla \cdot \sigma(\theta)\rho\nabla$ is a topological isomorphism between $W^{1,\mathfrak{q}}_{\Gamma_D}$ and $W^{-1,\mathfrak{q}}_{\Gamma_D}$ with

$$\sup_{\theta \in \mathcal{K}} \| \left(-\nabla \cdot \sigma(\theta)\rho\nabla\right)^{-1} \|_{L^\infty(J;\mathcal{L}(W^{-1,\mathfrak{q}}_{\Gamma_D};W^{1,\mathfrak{q}}_{\Gamma_D}))} < \infty. \tag{4.2}$$

Hence, for every $u \in \mathcal{U}$ there exists a unique $\psi = \psi_u \in L^{2s}(J;W^{1,\mathfrak{q}}_{\Gamma_D})$ such that

$$\psi_u(t) = \left(-\nabla \cdot \sigma(\theta_u(t))\rho\nabla\right)^{-1} u(t) \quad \text{in } W^{1,\mathfrak{q}}_{\Gamma_D}$$

for almost every $t \in (T_0, T_1)$, and

$$\sup_{u \in \mathcal{U}} \|\psi_u\|_{L^{2s}(J;W^{1,\mathfrak{q}}_{\Gamma_D})} < \infty.$$

Since $W^{1,q}_{\Gamma_D} \hookrightarrow W^{1,\mathfrak{q}}_{\Gamma_D}$ and, by uniqueness of ψ_u, we in particular obtain $\varphi_u = \psi_u$, such that the family φ_u is bounded in $L^{2s}(J;W^{1,\mathfrak{q}}_{\Gamma_D})$ as well. Estimating as in Lemma 3.27, we find that also

$$\sup_{u \in \mathcal{U}} \|(\sigma(\theta_u)\rho\nabla\varphi_u)\cdot\nabla\varphi_u\|_{L^s(J;L^{\mathfrak{q}/2})} < \infty.$$

Using the boundedness assumption on $\mathcal{K}$ in $L^s(J;W^{1,q})$, both the family of functionals $\tilde{\alpha}\theta_u$ and, here also employing boundedness of η, the divergence-operators $-\nabla \cdot \eta(\theta_u)\kappa\nabla\theta_u$ are uniformly bounded over $\mathcal{U}$, i.e.,

$$\sup_{u \in \mathcal{U}} \|\nabla \cdot \eta(\theta_u)\kappa\nabla\theta_u\|_{L^s(J;W_\emptyset^{-1,q})} + \|\tilde{\alpha}\theta_u\|_{L^s(J;W_\emptyset^{-1,q})} < \infty.$$

Sobolev embeddings give the embedding $L^{\mathfrak{q}/2} \hookrightarrow W_\emptyset^{-1,\varsigma}$ for $\varsigma = \frac{3\mathfrak{q}}{6-\mathfrak{q}}$, and certainly $W_\emptyset^{-1,q} \hookrightarrow W_\emptyset^{-1,\varsigma}$ due to $q > \varsigma$. Hence,

$$\partial_t\theta_u = \nabla \cdot \eta(\theta_u)\kappa\nabla\theta_u - \tilde{\alpha}\theta_u + (\sigma(\theta_u)\rho\nabla\varphi_u)\cdot\nabla\varphi_u + \alpha\theta_l$$

is uniformly bounded over $\mathcal{U}$ in $L^s(J; W_\emptyset^{-1,\varsigma})$. This shows that $\mathcal{K}$ is bounded in the space $W^{1,s}(J; W_\emptyset^{-1,\varsigma}) \cap L^s(J; W^{1,q})$. By Corollary 3.20, $\mathcal{K}$ is then also bounded in a Hölder space and thus a (relatively) compact set in $C(\overline{Q})$. This was the first claim. Now consider a sequence $(u_n) \subset \mathcal{U}$, converging in $L^{2s}(J; W_{\Gamma_D}^{-1,q})$ to the limit $\bar{u} \in \overline{\mathcal{U}}$. By compactness of $\mathcal{K}$, the sequence of associated solutions (θ_{u_n}) admits a subsequence which converges in $C(\overline{Q})$. But then (i) shows that $\bar{u} \in \mathcal{U}_g$, hence $\overline{\mathcal{U}} \subseteq \mathcal{U}_g$.

∎

Remark 4.8. *Note that we used Proposition 3.3 instead of Lemma 3.7 at the beginning of the proof of the second assertion in Theorem 4.7. This is indeed a crucial point, since Proposition 3.3 implies the isomorphism property and a* uniform *bound of the inverse of the elliptic divergence-gradient operator for all coefficient functions that share the same ellipticity constant and the same L^∞-bound. Thus, in our concrete situation, the norm of $(-\nabla \cdot \sigma(\theta)\rho\nabla)^{-1}$ is completely determined by $\Omega \cup \Gamma_D$ and the data from Assumption 2.6 (i) and 2.6 (ii), which gives the estimate in (4.2). By contrast, the application of Lemma 3.7 would require $\mathcal{K}$ in Theorem 4.7 to already be compact in the space of continuous functions to obtain a uniform bound, see also Corollary 3.24. This however is exactly the searched-for information, so that Proposition 3.3 is indeed essential for the proof of Theorem 4.7. Since the integrability exponent from Proposition 3.3 is in general less than 3 and therefore less than q, one needs an improved regularity in time to obtain the continuous embedding in the desired Hölder space, cf. Corollary 3.20. Therefore it is* not *sufficient to require $s > r^*(q)$ and the more restrictive condition $s > \bar{r}(q, \varsigma)$ is imposed instead.*

Next, we incorporate the control- and state constraints in (P) into the control problem. For this purpose, let us introduce the set

$$\mathcal{U}^{\mathrm{ad}} := \{u \in L^2(J; L^2(\Gamma_N)) \colon 0 \le u \le u_{\max} \text{ a.e. in } \Sigma_N\}. \tag{4.3}$$

Definition 4.9 (Feasible controls). *We call a global control $u \in \mathcal{U}_g$* feasible, *if $u \in \mathcal{U}^{ad}$ and the associated state satisfies $\mathcal{S}(u)(x,t) \le \theta_{\max}(x,t)$ for all $(x,t) \in \overline{Q}$.*

While the state constraints give upper bounds on the values of feasible solutions, lower bounds are natural in the problem and implicitly contained in (1.1)–(1.6) in the sense that the temperature of the workpiece associated with Ω will not drop below the minima of the surrounding temperature (represented by θ_l) and the initial temperature distribution θ_0.

Lemma 4.10 (Lower bounds for θ). *For every solution (θ, φ) in the sense of Theorem 3.14 with maximal existence interval $J_{\max}$, we have $\theta(x,t) \ge m_{\inf} := \min(\operatorname{ess\,inf}_\Sigma \theta_l, \min_{\overline{\Omega}} \theta_0)$ for all $(x,t) \in \overline{\Omega} \times [T_0, T_\bullet]$, where $T_\bullet \in J_{\max}$.*

See Lemma A.2 in the Appendix for a proof. Analogously, we find that $u \equiv 0$ is a feasible control under Assumption 2.8 (iv), the latter demanding that the surrounding temperature and the initial temperature do not exceed the state bounds at any point.

Corollary 4.11 (Nonempty feasible set). *The zero control $u \equiv 0$ is a feasible one.*

Proof. By Corollary 4.4, $u \equiv 0$ is a global control corresponding to $\varphi \equiv 0$. It obviously satisfies the control constraints, and using the same reasoning as in Lemma A.2 with Assumption 2.8 (iv), we obtain $\theta_{u \equiv 0} \le \theta_{\max}$. ∎

Let us next introduce a modified control space, fitting the norm in the objective functional in (P). So far, the controls originated from the space $L^{2s}(J; W_{\Gamma_D}^{-1,q})$ with $s > \bar{r}(q, \varsigma)$. For the optimization, we now switch to the more advanced control space

$$\mathbb{U} := W^{1,2}(J; L^2(\Gamma_N)) \cap L^p(J; L^p(\Gamma_N)) \tag{4.4}$$

with the standard norm $\|u\|_{\mathbb{U}} = \|u\|_{W^{1,2}(J;L^2(\Gamma_N))} + \|u\|_{L^p(J;L^p(\Gamma_N))}$. Since $p > \frac{4}{3}q - 2$ by Assumption 2.8, this space continuously embeds into $L^{2s}(J; W^{-1,q}_{\Gamma_D})$, which will give the boundedness required in Theorem 4.7 for a bounded set in $\mathcal{U}$. Moreover, this embedding is even *compact*, as the following result shows:

LEMMA 4.12. *Let $p > 2$. Then $\mathbb{U}$ is embedded into a Hölder space $C^{\varrho}(\overline{J}; L^{\mathfrak{p}}(\Gamma_N))$ for some $\varrho > 0$ and $2 < \mathfrak{p} < \frac{p+2}{2}$. In particular, there exists a* compact *embedding $\mathcal{E} \colon \mathbb{U} \hookrightarrow L^s(J; W^{-1,q}_{\Gamma_D})$ for every $p > \frac{4}{3}q - 2$ and $s \in [1, \infty]$.*

The proof can be found in the appendix, cf. Lemma A.3.

DEFINITION 4.13 (Reduced optimal control problem). *Consider the embedding $\mathcal{E}$ from Lemma 4.12 with range in $L^{2s}(J; W^{-1,q}_{\Gamma_D})$, where $s > \bar{r}(q, \varsigma)$ is the integrability exponent from the objective functional. We set*

$$\mathbb{U}_g := \{u \in \mathbb{U} \colon \mathcal{E}(u) \in \mathcal{U}_g\}$$

and define the mapping

$$\mathcal{S}_{\mathcal{E}} := \mathcal{S} \circ \mathcal{E} \colon \mathbb{U}_g \to W^{1,s}(J; W^{-1,q}_{\emptyset}) \cap L^s(J; W^{1,q}).$$

Moreover, we define the reduced objective functional *j obtained by reducing the objective functional in* (P) *to u, i.e.,*

$$j(u) = \frac{1}{2} \int_E |\mathcal{S}_{\mathcal{E}}(u)(T_1) - \theta_d|^2 \,\mathrm{d}x + \frac{\gamma}{s} \|\nabla \mathcal{S}_{\mathcal{E}}(u)\|^s_{L^s(J;L^q)} + \frac{\beta}{2} \int_{\Sigma_N} (\partial_t u)^2 + |u|^p \,\mathrm{d}\omega \,\mathrm{d}t,$$

as a function on $\mathbb{U}_g$. Further, let $\mathbb{U}^{ad} := \mathbb{U} \cap \mathcal{U}^{ad}$ and $\mathbb{U}^{ad}_g := \mathbb{U}_g \cap \mathcal{U}^{ad}$, where $\mathcal{U}^{ad}$ is as defined in (4.3).

The following is the main result for this section:

THEOREM 4.14 (Existence of optimal controls). *There exists an optimal solution $\bar{u} \in \mathbb{U}^{ad}_g$ to the problem*

$$\min_{u \in \mathbb{U}^{ad}_g} j(u) \quad \textit{such that} \quad \mathcal{S}_{\mathcal{E}}(u)(x,t) \le \theta_{\max}(x,t) \quad \forall (x,t) \in \overline{Q}.$$

Proof. Thanks to the existence of the feasible control $u \equiv 0$, cf. Corollary 4.11, the objective functional is bounded from below by 0. Thus there exists a minimizing sequence of feasible controls (u_n) in $\mathbb{U}^{\mathrm{ad}}_g$ such that $j(u_n) \to \inf_{u \in \mathbb{U}^{\mathrm{ad}}_g} j(u)$ in $\mathbb{R}$. On account of

$$\int_{\Sigma_N} (\partial_t u)^2 + |u|^p \,\mathrm{d}\omega \,\mathrm{d}t \longrightarrow \infty \quad \text{when} \quad \|u\|_{\mathbb{U}} \longrightarrow \infty, \tag{4.5}$$

the objective functional is radially unbounded so that the minimizing sequence is bounded in $\mathbb{U}$ and, due to reflexivity of $\mathbb{U}$, has a weakly convergent subsequence (again (u_n)), converging weakly to some $\bar{u} \in \mathbb{U}$. As $\mathbb{U}^{\mathrm{ad}}$ is closed and convex, we have $\bar{u} \in \mathbb{U}^{\mathrm{ad}}$. By the compact embedding from Lemma 4.12, (u_n) converges strongly in $L^{2s}(J; W^{-1,q}_{\Gamma_D})$, also to $\bar{u} \in L^{2s}(J; W^{-1,q}_{\Gamma_D})$. The fact that state constraints are present and Lemma 4.10 imply that the family (θ_{u_n}) is uniformly bounded in time and space for every feasible control u. Together with the gradient term in the objective functional, Theorem 4.7 (ii) now shows that (θ_{u_n}) admits a subsequence which converges in $C(\overline{Q})$. By Theorem 4.7 (i) in turn, this means $\bar{u} \in \mathbb{U}_g$, hence $\bar{u} \in \mathbb{U}^{\mathrm{ad}}_g$. Now another application of [48, Lem. 5.5] shows that $\mathcal{S}_{\mathcal{E}}(u_n) \to \mathcal{S}_{\mathcal{E}}(\bar{u})$ in

$W^{1,s}(J;W_\emptyset^{-1,q}) \cap L^s(J;W^{1,q})$ after switching to the appropriate subsequence, which immediately implies convergence of the first two terms in the objective functional for this subsequence (each as n goes to infinity). The third term, corresponding to $\mathbb{U}$, is clearly continuous and convex on $\mathbb{U}$ and as such weakly lower semicontinuous, hence we find

$$\inf_{u\in\mathbb{U}_g^{\mathrm{ad}}} j(u) = \lim_{n\to\infty} j(u_n) \geq j(\bar{u})$$

and thus $j(\bar{u}) = \inf_{u\in\mathbb{U}_g^{\mathrm{ad}}} j(u)$. ∎

REMARK 4.15. *In the proof of Theorem 4.14, boundedness of the minimizing sequence (u_n) in the control space $\mathbb{U}$ was essential and followed from the radial unboundedness of the objective functional as seen in (4.5). Alternatively, one could also assume that the upper bound $u_{\max}$ in the control constraints satisfies $u_{\max} \in L^p(J;L^p(\Gamma_N))$ with $p > \frac{4}{3}q - 2$. In this case, an objective functional of the form*

$$\frac{1}{2}\|\theta(T_1) - \theta_d\|^2_{L^2(E)} + \frac{\gamma}{s}\|\nabla\theta\|^s_{L^s(T_0,T_1;L^q(\Omega))} + \frac{\beta}{2}\int_{\Sigma_N} (\partial_t u)^2 \,\mathrm{d}\omega\,\mathrm{d}t$$

is sufficient to establish the existence of a globally optimal control.

So far, we were able to show that there exists an optimal global solution to (P) by using the properties of feasible control functions and their associated solutions to the PDE system induced by the objective functional. In [47], we further show that the set of global solutions $\mathcal{U}_g$ is in fact open and use this property to derive necessary optimality conditions of first order for (P).

Appendix A. Proofs of auxiliary results.

LEMMA A.1. *Let $q \in (3,4)$ and $\varsigma \in [2,q]$. For every $\tau \in (0, \frac{q-3}{2q}(1-\frac{3}{q}+\frac{3}{\varsigma})^{-1})$, the interpolation space $(W^{1,q}, W_\emptyset^{-1,\varsigma})_{\tau,1}$ embeds into some Hölder space $C^\delta(\overline{\Omega})$ with $\delta > 0$. If $\Omega \cup \Gamma_D$ even satisfies the volume-conservation condition from Definition 2.3 (ii), then we can additionally admit $\varsigma \in (1,2)$.*

Proof. We apply the reiteration theorem [54, Ch. 1.10.2] and a general interpolation principle to obtain

$$\begin{aligned}(W^{1,q}, W_\emptyset^{-1,\varsigma})_{\tau,1} &= (W^{1,q}, (W^{1,q}, W_\emptyset^{-1,\varsigma})_{\frac{1}{2},1})_{2\tau,1} \\ &\hookrightarrow (W^{1,q}, (W^{1,\varsigma}, W_\emptyset^{-1,\varsigma})_{\frac{1}{2},1})_{2\tau,1}. \end{aligned} \tag{A.1}$$

We next show that $(W^{1,\varsigma}, W_\emptyset^{-1,\varsigma})_{\frac{1}{2},1} \hookrightarrow L^\varsigma$. If $\Omega \cup \Gamma_D$ satisfies the volume-preserving condition, then this follows from [23, Lem. 3.4] and [54, Ch. 1.10.3, Thm. 1]. Otherwise, for $\varsigma \in [2,q]$, let $\mathcal{D}_\varsigma(\mathrm{id}_3)$ denote the domain of the Laplacian $-\Delta + 1$ acting on the Banach space $W_\emptyset^{-1,\varsigma}$, cf. also Remark 3.2. We use two facts about this operator, proven in [7, Thm. 5.1/11.5]:

(i) It is a positive operator in the sense of [54, Ch. 1.14]. In particular, its fractional powers are well-defined and, denoting the domain of $(-\Delta+1)^{1/2}$ on $W_\emptyset^{-1,\varsigma}$ by $\mathcal{D}_\varsigma(\mathrm{id}_3)^{\frac{1}{2}}$, one has $(W_\emptyset^{-1,\varsigma}, \mathcal{D}_\varsigma(\mathrm{id}_3))_{\frac{1}{2},1} \hookrightarrow \mathcal{D}_\varsigma(\mathrm{id}_3)^{\frac{1}{2}}$, cf. [54, Ch. 1.15.2].

(ii) The square root $(-\Delta+1)^{1/2}$ even satisfies $(-\Delta+1)^{1/2} \in \mathcal{LH}(W_\emptyset^{-1,\varsigma}; L^\varsigma)$, or in other words, $\mathcal{D}_\varsigma(\mathrm{id}_3)^{\frac{1}{2}} = L^\varsigma$.

Observing $W^{1,\varsigma} \hookrightarrow \mathcal{D}_\varsigma(\mathrm{id}_3)$, we now find

$$(W^{1,\varsigma}, W_\emptyset^{-1,\varsigma})_{\frac{1}{2},1} \hookrightarrow (W_\emptyset^{-1,\varsigma}, \mathcal{D}_\varsigma(\mathrm{id}_3))_{\frac{1}{2},1} \hookrightarrow L^\varsigma.$$

For both cases, re-inserting into (A.1) yields $(W^{1,q}, W_{\emptyset}^{-1,\varsigma})_{\tau,1} \hookrightarrow (W^{1,q}, L^{\varsigma})_{2\tau,1}$.

To see that this space indeed embeds into a Hölder space, we define $p := \big(\frac{1-2\tau}{q} + \frac{2\tau}{\varsigma}\big)^{-1}$ and observe that $\delta := 1 - 2\tau - \frac{3}{p} \in (0,1)$, due to our condition on τ. Denoting by $H^{t,p}$ the corresponding space of Bessel potentials (cf. [54, Ch. 4.2.1]) one has the embedding $H^{1-2\tau,p} \hookrightarrow C^{\delta}(\overline{\Omega})$, see [54, Thm. 4.6.1]. This, combined with the interpolation inequality for $H^{1-2\tau,p}$ ([23, Thm. 3.1]) gives for any $\psi \in W^{1,q}$ the estimate

$$\|\psi\|_{C^{\delta}(\overline{\Omega})} \le \|\psi\|_{H^{1-2\tau,p}} \le \|\psi\|_{W^{1,q}}^{1-2\tau} \|\psi\|_{L^{\varsigma}}^{2\tau}. \tag{A.2}$$

But it is well-known (cf. [54, Ch. 1.10.1] or [9, Ch. 5, Prop. 2.10]) that an inequality of type (A.2) is constitutive for the embedding $(W^{1,q}, L^{\varsigma})_{2\tau,1} \hookrightarrow C^{\delta}(\overline{\Omega})$. ∎

Lemma A.2. *For every solution (θ,φ) in the sense of Theorem 3.14 with maximal existence interval $J_{\max}$, it is true that $\theta(x,t) \ge \min(\operatorname{ess\,inf}_{\Sigma} \theta_l, \min_{\overline{\Omega}} \theta_0)$ for all $(x,t) \in \overline{\Omega} \times [T_0, T_\bullet]$, where $T_\bullet \in J_{\max}$.*

Proof. We set $m_{\inf} := \min(\operatorname{ess\,inf}_{\Sigma} \theta_l, \min_{\overline{\Omega}} \theta_0)$ and $\zeta(t) = \theta(t) - m_{\inf}$ and decompose $\zeta(t)$ into its positive and negative part, that is, $\zeta(t) = \zeta^+(t) - \zeta^-(t)$ with both $\zeta^+(t)$ and $\zeta^-(t)$ being positive functions. By [15, Ch. IV, §7, Prop. 6/Rem. 12] we then have that $\zeta^-(t)$ is still an element of $W^{1,q}$ for almost every $t \in (T_0, T_\bullet)$. In particular, we may test (3.5) against $-\zeta^-(t)$, insert $\theta = \zeta + m_{\inf}$ and use that $m_{\inf}$ is constant:

$$\begin{aligned} -\int_\Omega \partial_t \zeta(t)\zeta^-(t)\,\mathrm{d}x - \int_\Omega (\eta(\theta(t))\kappa\nabla\zeta(t)) \cdot \nabla\zeta^-(t)\,\mathrm{d}x - \int_\Gamma \alpha\zeta(t)\zeta^-(t)\,\mathrm{d}x \\ = -\int_\Gamma \alpha(\theta_l(t) - m_{\inf})\zeta^-(t) - \int_\Omega \zeta^-(t)(\sigma(\theta(t))\rho\nabla\varphi(t)) \cdot \nabla\varphi(t)\,\mathrm{d}x. \end{aligned}$$

Observe that the support of products of $\zeta(t)$ and $\zeta^-(t)$ is exactly the support of $\zeta^-(t)$, and $\zeta(t) = -\zeta^-(t)$ there. We thus obtain (see [55])

$$\begin{aligned} \frac{1}{2}\,\partial_t \big\|\zeta^-(t)\big\|_{L^2}^2 + \int_\Omega (\eta(\theta(t))\kappa\nabla\zeta^-(t)) \cdot \nabla\zeta^-(t)\,\mathrm{d}x + \int_\Gamma \alpha\zeta^-(t)^2\,\mathrm{d}x \\ = -\int_\Gamma \alpha(\theta_l(t) - m_{\inf})\zeta^-(t) - \int_\Omega \zeta^-(t)(\sigma(\theta(t))\rho\nabla\varphi(t)) \cdot \nabla\varphi(t)\,\mathrm{d}x. \end{aligned} \tag{A.3}$$

Let us show that $\partial_t\|\zeta^-(t)\|_{L^2}^2 \le 0$. By Assumption 2.6, $(\eta(\theta(t))\kappa\nabla\zeta^-(t)) \cdot \nabla\zeta^-(t) \ge \underline{\eta}\underline{\kappa}\|\nabla\zeta^-(t)\|_{\mathbb{R}^3}^2$ and $-(\sigma(\theta(t))\rho\nabla\varphi(t))\cdot\nabla\varphi(t) \le -\underline{\sigma}\underline{\rho}\|\nabla\varphi(t)\|_{\mathbb{R}^3}^2$. This means that both integrals on the left-hand side in (A.3) are positive (since $\alpha \ge 0$), while the second term on the right-hand side is negative. The constant $m_{\inf}$ is constructed exactly such that $\theta_l(t) - m_{\inf}$ is greater or equal than zero almost everywhere, such that $-\alpha(\theta_l(t) - m_{\inf})\zeta^-(t) \le 0$. Hence, from (A.3) it follows that $\partial_t\|\zeta^-(t)\|_{L^2}^2 \le 0$. But, due to the construction of ζ, we have $\zeta(T_0) \ge 0$, which means that $\zeta^-(T_0) \equiv 0$ and thus $\zeta^-(t) \equiv 0$ for all $t \in (T_0, T_\bullet)$. ∎

Lemma A.3. *Let $p > 2$. Then the space $\mathbb{U}$ is embedded into a Hölder space $C^{\varrho}(\overline{J}; L^{\mathfrak{p}}(\Gamma_N))$ for some $\varrho > 0$ and $2 < \mathfrak{p} < \frac{p+2}{2}$. In particular, there exists a* compact *embedding $\mathcal{E}\colon \mathbb{U} \hookrightarrow L^s(J; W_{\Gamma_D}^{-1,q})$ for every $p > \frac{4}{3}q - 2$ and $s \in [1,\infty]$.*

Proof. From the construction of real interpolation spaces by means of the trace method it immediately follows that

$$\mathbb{U} \hookrightarrow C(\overline{J}; (L^p(\Gamma_N), L^2(\Gamma_N))_{\frac{2}{p+2},\frac{p+2}{2}}) = C(\overline{J}; L^{\frac{p+2}{2}}(\Gamma_N)),$$

see [54, Ch. 1.8.1–1.8.3 and Ch. 1.18.4]. With similar reasoning as for (3.10), see also [34, Lem. 3.17] and its proof, we also may show $\mathbb{U} \hookrightarrow C^{\varrho}(\overline{J}; (L^p(\Gamma_N), L^2(\Gamma_N))_{\tau,1})$ for all $\tau \in (\frac{2}{2+p}, 1)$ and some $\varrho = \varrho(\tau) > 0$. Moreover,

$$\big(L^p(\Gamma_N), L^2(\Gamma_N)\big)_{\tau,1} \hookrightarrow \big[L^p(\Gamma_N), L^2(\Gamma_N)\big]_\tau = L^{\mathfrak{p}}(\Gamma_N)$$

with $\mathfrak{p} = \mathfrak{p}(\tau) = (\frac{1-\tau}{p} + \frac{\tau}{2})^{-1} \in (2, \frac{2+p}{2})$ for $\tau \in (\frac{2}{2+p}, 1)$, see [54, Ch. 1.10.1/3 and Ch. 1.18.4]. This means we have $\mathbb{U} \hookrightarrow C^{\varrho}(\overline{J}; L^{\mathfrak{p}}(\Gamma_N))$ for all $\mathfrak{p} \in (2, \frac{2+p}{2})$, with $\varrho > 0$ depending on $\mathfrak{p}$. If $\mathfrak{p} > \frac{2}{3}q$, then there is an embedding $L^{\mathfrak{p}}(\Gamma_N)) \hookrightarrow W^{-1,q}_{\Gamma_D}$, cf. Remark 2.7, and this is even compact in this case as we will show below. To make $\mathfrak{p} > \frac{3}{2}q$ possible, we need $\frac{p+2}{2} > \frac{2}{3}q$, which is equivalent to $p > \frac{4}{3}q - 2$. Now the vector-valued Arzelà-Ascoli Theorem, cf. [43, Thm. 3.1], yields the assertion.

It remains to show that $L^{\mathfrak{p}}(\Gamma_N) \hookrightarrow W^{-1,q}_{\Gamma_D}$ compactly for $\mathfrak{p} > \frac{2}{3}q$, or equivalently $W^{1,q'}_{\Gamma_D} \hookrightarrow L^{\mathfrak{p}'}(\Gamma_N)$ compactly. This follows from $W^{1,q'} \hookrightarrow L^{p'}(\partial\Omega)$ compactly, which is proven in the main result in [10]. The assumptions there are satisfied because firstly, cf. Remark 2.2, Ω admits a $W^{1,q}$-extension operator and is thus a 3-set in the sense of Jonsson and Wallin [38], see [29]. Secondly, the Lipschitz property of the boundary implies that $\partial\Omega$ is a 2-set as explained in [38, Ch. II, Ex. 1] from which the measure condition follows. ∎

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